A Unified Framework for Monetary Theory and Policy Analysis*

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September 10, 2002

Abstract

Search-theoretic models of monetary exchange are based on explicit descriptions of the frictions that make money essential. However, tractable versions usually have strong assumptions that make them ill-suited for discussing some policy questions, especially those concerning changes in the money supply. Hence most policy analysis uses reduced-form models. We propose a framework that attempts to bridge this gap: it is based explicitly on microeconomic frictions, but allows for interesting macroeconomic policy analyses. At the same time, the model is analytically tractable and amenable to quantitative analysis.

*First version: May 2000. Many people provided extremely helpful input to this project, including S.B. Aruoba, A. Berentsen, V.V. Chari, L. Christiano, N. Kocherlakota, G. Rocheteau, S. Shi and N. Wallace. Financial support from the C.V. Starr Center for Applied Economics at NYU, STICERD at the LSE, the NSF, and the Federal Reserve Bank of Cleveland is acknowledged. The usual disclaimer applies.

"The matching models are without doubt ingenious and beautiful. But it is quite hard to integrate them with the rest of macroeconomic theory – not least because they jettison the basic tool of our trade, competitive markets."

Kiyotaki and Moore (2001)

1 Introduction

This paper is an attempt to provide a unification, or at least to develop some common ground, between micro and macro models of monetary exchange. Why? First, existing macro models are all to some extent reduced-form models. By this we simply mean they make assumptions, such as putting money in the utility function or imposing cash-in-advance constraints, that are presumably meant to stand in for some role for money that is not made explicit but ought to be – say, that it helps overcome spatial, temporal, or informational frictions. Second, attempts to provide micro foundations for monetary economics using search theory, with explicit descriptions of specialization, the pattern of meetings, the information structure, and so on, typically need very strong assumptions for tractability. For example, there are often extreme restrictions on how much money agents can hold, and this makes the analyses of some policy issues difficult at best.[1]

We have several goals. We want a framework that, like existing macro models, allows one to analyze standard issues in monetary economics in both a qualitative and quantitative fashion; an example is to determine the welfare

[1] In terms of the literature, the reduced form approach is vast, but examples include Lucas and Stokey (1983, 1987), Cooley and Hansen (1989) and Christiano et al. (1997); see Walsh (1998) for other references. We go into more detail below on the search literature, but examples include Kiyotaki and Wright (1989, 1991), Shi (1995), Trejos and Wright (1995), Kocherlakota (1998) and Wallace (2001).

cost of inflation. At the same time we want a model where the role for money is explicit so that we can to address some issues that are studied more naturally with search-based than reduced-form models; examples include, to ask exactly what frictions make the use of money an equilibrium or an efficient arrangement, and to show how different regimes lead to different outcomes (say, commodity versus fiat money). Finally, we want the framework to be tractable and capable of delivering clean analytic results, but at the same time we want that it will be relatively easily and realistically calibrated.

There are of course previous attempts to provide search-based monetary models without the severe restrictions on money holdings. Trejos and Wright (1995) present a general version of their model where agents can hold any $m \in \mathbb{R}_+$ but cannot solve it, and resort to assuming $m \in \{0, 1\}$. The model with $m \in \mathbb{R}_+$ was studied numerically by Molico (1999). Although his findings are interesting, unfortunately the framework is quite complicated. Not many results are available, except those found by computation, and even numerically the model is difficult to analyze. And numerical methods are not especially useful for looking at existence, multiplicity, dynamics, and a host of other issues that are important in monetary economics. One of the main problems with the model is the endogenous distribution of money holdings across agents, $F(m)$, is nondegenerate, and hence the model has a built-in heterogeneity that is hard to handle analytically.

The approach pioneered by Shi (1997) gets around the problem by making some creative assumptions to render $F(m)$ degenerate. In our model $F(m)$ will also be degenerate, although the economic and technical details of the model will differ significantly. We will have a lot to say about the comparison between our framework and various alternatives later. Here we simply mention that at the end of the day some but not all of our results will look similar to previous search models, and some even look a lot like what comes out of reduced form models – which is as it should be, since those models are meant to be descriptive of what one sees in actual economies. At the same time, it is clearly desirable to have solid micro foundations, and making these explicit not only leads to new insights, it can also change the quantitative answers to some basic economic issues, like the welfare cost of inflation.

The rest of the paper is organized as follows. Section 2 presents the basic model, defines equilibrium, gives the main results, and compares our framework to the related literature. Section 3 presents extensions and a discussion of monetary policy, including a fully calibrated version of the model. Section 4 concludes. Many technical results are contained in the Appendix.

2 The Basic Model

2.1 Environment

Time is discrete. There is a $[0, 1]$ continuum of agents who live forever and have discount factor $\beta \in (0, 1)$. In the interest of integrating standard macro and search models, we assume there are two types of commodities: *general* and *special* goods. As in standard macro models, all agents consume and produce general goods. The utility of consuming Q units of the general good is $U(Q)$ and the disutility of producing Q units is $C(Q)$. Here agents produce this good themselves, but one can recast things by letting them supply labor h at disutility $C(h)$, and have firms convert this into general goods via a standard production function (see Aruoba and Wright [2002]).

It is important that either U or C is linear. We assume here $C(Q) = Q$, and that U is C^2 (twice continuously differentiable) with $U' > 0$ and $U'' \leq 0$. Also, U is either unbounded or at least satisfies a condition in Lemma 6, and $U'(X^*) = 1$ for $X^* \in (0, \infty)$ with $U(X^*) > X^*$.

In contrast to general goods, each agent produces a subset and consumes a subset of special commodities, as in search models. In particular, given two agents i and j drawn at random there are four possible events. The probability that both consume something the other can produce (a double coincidence) is δ. The probability that i consumes something j produces but not vice-versa (a single coincidence) is σ. Symmetrically, the probability that j consumes something i produces but not vice-versa is also σ. And the probability that neither wants anything the other can produce is $1 - 2\sigma - \delta$, where $2\sigma \leq 1 - \delta$. This notation captures several explicit specifications for specialization in the literature as special cases.[2] In a single coincidence meeting, if i wants what j produces we call i the *buyer* and j the *seller*.

Let $u(q)$ be the utility of consumption and $c(q)$ the disutility production of any special good, where u and c are C^n with $n > 2$. We assume $u(0) = c(0) = 0$, $u'(q) > 0$, $c'(q) > 0$, $u''(q) < 0$, $c''(q) \geq 0$, and $u(\bar{q}) = c(\bar{q})$ for some $\bar{q} > 0$. We use q^* to denote the efficient quantity of special good production, which solves $u'(q^*) = c'(q^*)$; q^* is what all agents would agree to ex ante if they had some way of committing to or enforcing the agreement. Note that we can always normalize $c(q) = q$, without loss in generality, as long

[2] For example, in Kiyotaki and Wright (1989) or Aiyagari and Wallace (1991) there are N goods and N types, where type n produces good n and consumes good $n+1$ (mod N). If $N > 2$ we have $\sigma = 1/N$ and $\delta = 0$, while if $N = 2$ we have $\delta = 1/2$ and $\sigma = 0$. In Kiyotaki and Wright (1993), the event that i consumes what j produces is independent of the event that j consumes what i produces, and each occurs with probability x. Then $\delta = x^2$ and $\sigma = x(1-x)$.

as we re-scale $u(q)$; this merely amounts to measuring output in utils rather than physical units. At one point we use a condition on the third derivative, $u''' \leq (u'')^2/u'$; a simple way to state this is to say that marginal utility is *log-concave* – i.e., $\log u'$ is concave.

General and special goods are nonstorable, but there is another object called *money* that can be stored. Money, like goods, is perfectly divisible and agents can hold any quantity $m \geq 0$. Money has no intrinsic value but could potentially be used in trade – although it is not *necessary* to use money; it is generally possible, e.g., to barter special goods directly. One cannot trade special for general goods, however, due to the following assumption: in each period there are two sub-periods, day and night, and special goods are only produced during the day while general goods are only produced at night. Given these goods are nonstorable, the only feasible trades during the day are barter in special goods or the exchange of special goods for money, and the only feasible trades at night are barter in general goods or the exchange of general goods for money.

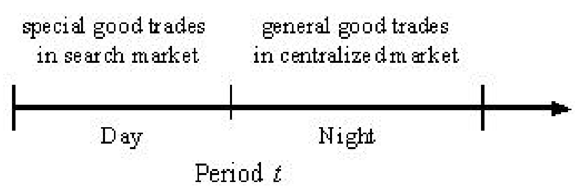

Figure 1: Timing.

During the day agents participate in a bilateral matching process, as in standard search theory. In this decentralized market there is a probability α of a meeting each period, each meeting is a random draw from the population, and the terms of trade are determined by bargaining. At night there is a frictionless centralized market where one dollar buys ϕ units of general goods – i.e., $p_g = 1/\phi$ is the nominal price, and agents take it parametrically. The timing is illustrated in Figure 1. All trade in the decentralized market must be quid pro quo, either goods for goods or goods for money; there is no credit, because the matching process is anonymous and hence there is no punishment for reneging on debt (Kocherlakota [1998]; Wallace [2001, 2002]). We could allow intertemporal trade in general goods, but in equilibrium it will not happen, at least in the basic model with no intrinsic heterogeneity.[3]

2.2 Equilibrium

In this subsection we build gradually towards the definition of equilibrium. We begin by describing the value functions, taking as given the terms of trade and the distribution of money. In general, the state variable for an individual includes his own money holdings m and a vector of aggregate states s. At this point we let $s = (\phi, F)$, where ϕ is the value of money in the centralized market and F is the distribution of money holdings in the decentralized market – i.e., $F(\tilde{m})$ is the measure of agents in this market holding $m \leq \tilde{m}$. Necessarily F satisfies $\int m dF(m) = M$ at every date, where M is the total money stock that is fixed for now (but see below). The agent takes as given

[3] We could price a bond, say, but it will not trade since we cannot find one agent who wants to save and another who wants to borrow at the same interest rate. In a generalized version of the model agents may want to engage in intertemporal trade. However, one could rule it out by assuming agents are anonymous in the centralized market. There is, of course, nothing inconsistent with anonymity and centralized trading; indeed, there is a long tradition of interpreting competitive markets as anonymous. Wallace (2002) discusses this further, and offers Levine (1991) as an example of a model with centralized trading among anonymous agents where money is essential. Finally, we want to mention that what is important here is that there are two types of markets – centralized and decentralized – and *not* that there are different goods: everything goes through if there are no general goods and special goods are traded in both markets.

a law of motion $s_{+1} = \Upsilon(s)$, but it will be determined in equilibrium.[4]

Let $V(m, s)$ be the value function for an agent with m dollars in the morning when he enters the decentralized market, and $W(m, s)$ the value function in the afternoon when he enters the centralized market, given s. Let $q(m, \tilde{m}, s)$ and $d(m, \tilde{m}, s)$ be the quantity of goods and dollars that change hands in a single coincidence meeting between a buyer with m and a seller with $\tilde{m}$ dollars. Let $B(m, \tilde{m}, s)$ be the payoff for an agent with m who meets an agent with $\tilde{m}$ when there is a double coincidence of wants. Bellman's equation is

$$\begin{aligned} V(m,s) &= \alpha\sigma \int \{u[q(m,\tilde{m},s)] + W[m - d(m,\tilde{m},s)]\} dF(\tilde{m}) \\ &+ \alpha\sigma \int \{-c[q(\tilde{m},m,s)] + W[m + d(\tilde{m},m,s)]\} dF(\tilde{m}) \quad (1) \\ &+ \alpha\delta \int B(m,\tilde{m},s) dF(\tilde{m}) + (1 - 2\alpha\sigma - \alpha\delta) W(m,s). \end{aligned}$$

The first term is the expected payoff from a single coincidence meeting where you buy $q(m, \tilde{m}, s)$ and then go to the centralized market with $m - d(m, \tilde{m}, s)$ dollars. Other terms have similar interpretations.

The value of entering the centralized market with m dollars is

$$\begin{aligned} W(m,s) &= \max_{X,Y,m_{+1}} \{U(X) - Y + \beta V(m_{+1}, s_{+1})\} \\ s.t.\ X &= Y + \phi m - \phi m_{+1} \end{aligned}$$

where X is consumption and Y production of general goods, and m_{+1} is money taken out of this market. We impose $X \geq 0$ and $m_{+1} \geq 0$, but we do

[4] The "trick" of putting ϕ in the state vector allows us to capture nonstationary equilibria while still using recursive methods; Duffie et al. (1994) use a similar approach in an overlapping generations model. In any case, for much of this paper we focus on steady states where ϕ (or some transformation, like ϕM) is constant and the point is moot. For now, we are not requiring anything to be stationary, but we omit the t subscript when there is no risk of confusion; e.g., at t we write the current state as s and next period's state as s_{+1}.

not impose $Y \geq 0$. Rather, our approach is to allow any $Y \in \mathbb{R}$ for now, and then after finding equilibrium we can impose conditions to rule out $Y < 0$. Substituting for Y,

$$W(m,s) = U(X^*) - X^* + \phi m + \max_{m_{+1}} \{-\phi m_{+1} + \beta V(m_{+1}, s_{+1})\} \tag{2}$$

where $U'(X^*) = 1$. This immediately implies the choice m_{+1} does not depend on m. Moreover, it implies W is linear (affine) in m:[5]

$$W(m,s) = W(0,s) + \phi m. \tag{3}$$

We now consider the terms of trade in the decentralized market, which are determined by bargaining. There are two bargaining situations to consider: single coincidence and double coincidence meetings. In the case of a double coincidence we adopt the symmetric Nash bargaining solution with the threat point of an agent given by his continuation value $W(m,s)$. Lemma 1 in the Appendix proves that, regardless of the money holdings of the two agents, this implies that in any double coincidence meeting the agents give each other the efficient quantity q^* and no money changes hands. Thus, $B(m,\tilde{m},s) = u(q^*) - c(q^*) + W(m,s)$.

Now consider bargaining in a single coincidence meeting when the buyer has m and the seller $\tilde{m}$ dollars. Here we use the generalized Nash solution where the buyer has bargaining power θ and threat points are given by

[5] Basically, it is the fact that utility over (X,Y) is quasi-linear that rules out wealth effects, which makes m_{+1} independent of m, and makes W linear, at least as long as we are not at a corner solution. This is why we do not impose $Y \geq 0$ for now. One could avoid the issue entirely by assuming $U(X) = X$ (utility is linear in X as well as Y), or more generally $U(X) = \bar{U}(X)$ for $X \leq X^*$ and $U(X) = \bar{U}(X^*) - X^* + X$ (it is asymtoptically linear), since with these preferences $Y \geq 0$ cannot bind. One can also simply say that $Y < 0$ is fine, and reinterpret the production of $Y < 0$ as the consumption of $-Y > 0$ (this works best when X and Y are different goods). In any case, we will soon provide conditions that guarantee $Y > 0$ in any equilibrium.

continuation values. That is, (q, d) maximizes

$$[u(q) + W(m-d, s) - W(m, s)]^{\theta} [-c(q) + W(\tilde{m}+d, s) - W(\tilde{m}, s)]^{1-\theta}$$

subject to $d \leq m$. By virtue of (3), this simplifies to

$$\max_{q,d} [u(q) - \phi d]^{\theta} [-c(q) + \phi d]^{1-\theta} \tag{4}$$

subject to $d \leq m$. The constraint simply says you cannot spend more money than you have. There are also two side conditions, $u(q) \geq \phi d$ and $c(q) \leq \phi d$, but they never bind here.

The solution (q, d) to (4) does not depend on $\tilde{m}$, and depends on m only if the constraint $d \leq m$ binds. Also, it depends on s only through ϕ, and indeed only through real balances $z = \phi m$. We abuse notation slightly and write $q(m, \tilde{m}, s) = q(m)$ and $d(m, \tilde{m}, s) = d(m)$ in what follows (the dependence on ϕ is implicit). Lemma 2 in the Appendix proves that the bargaining solution is

$$q = \begin{cases} \widehat{q}(m) & \text{if } m < m^* \\ q^* & \text{if } m \geq m^* \end{cases} \text{ and } d = \begin{cases} m & \text{if } m < m^* \\ m^* & \text{if } m \geq m^* \end{cases} \tag{5}$$

where $\widehat{q}(m)$ solves the first order condition from (4), which for future reference we write as

$$\phi m = \frac{\theta c(q) u'(q) + (1-\theta) u(q) c'(q)}{\theta u'(q) + (1-\theta) c'(q)} \equiv z(q), \tag{6}$$

and $m^* = z(q^*)/\phi$.

Hence, if real balances are at least ϕm^* the buyer gets q^*; otherwise he spends all his money and gets $\widehat{q}(m)$, which as we soon will verify is less than q^*. Since u and c are C^n the implicit function theorem implies that, for all

$m < m^*$, $\widehat{q}$ is C^{n-1} and from (6) we have $\widehat{q}' = \phi/z'(q)$. Inserting z' explicitly and simplifying,

$$\widehat{q}' = \frac{\phi[\theta u' + (1-\theta)c']^2}{u'c'[\theta u' + (1-\theta)c'] + \theta(1-\theta)(u-c)(u'c'' - c'u'')}. \tag{7}$$

Hence, $\widehat{q}' > 0$ for all $m < m^*$. It is easy to check $\lim_{m \to m^*} \widehat{q}(m) = q^*$, and so we conclude $\widehat{q}(m) < q^*$ for all $m < m^*$, as seen in Figure 2.

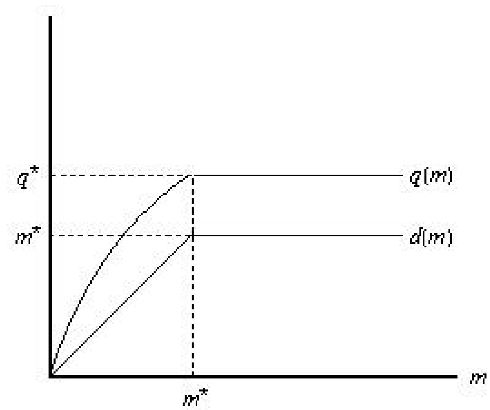

Figure 2: Single-coincidence bargaining solution.

We now insert the bargaining outcomes together with $W(m)$ into (1) and rewrite it as

$$V(m,s) = \max_{m_{+1}} \{v(m,s) + \phi m - \phi m_{+1} + \beta V(m_{+1}, s_{+1})\} \tag{8}$$

where

$$v(m,s) = v_0(s) + \alpha\sigma \{u[q(m)] - \phi d(m)\} \tag{9}$$

is a bounded and continuous function and $v_0(s)$ is independent of m and m_{+1}.[6] This not only gives us a convenient way to write Bellman's equation,

[6]In any equilibrium $v(m,s)$ is bounded and continuous for the following reason. First,

it allows us to establish that there exists a unique $V(m,s)$ in the relevant space of functions satisfying (8), even though this is a nonstandard dynamic programming problem (because V is unbounded in m due to the linear term ϕm).

We give the argument here for the case where s is constant – which does nothing to overcome the problem of unboundedness, but does simplify the presentation – and relegate the more general case to Lemma 7 in the Appendix. Given s is constant, write $V(m,s) = \hat{V}(m)$. Then consider the space of functions $\hat{V} : \mathbb{R}_+ \to \mathbb{R}$ that can be written $\hat{V}(m) = \hat{v}(m) + \phi m$ for some bounded and continuous function $\hat{v}(m)$. For any two functions in this space $\hat{V}_1(m) = \hat{v}_1(m) + \phi m$ and $\hat{V}_2(m) = \hat{v}_2(m) + \phi m$, we can define $\left\|\hat{V}_1 - \hat{V}_2\right\| = \sup_{m\in\mathbb{R}_+} |\hat{v}_1(m) - \hat{v}_2(m)|$, and this constitutes a complete metric space. One can show the right hand side of (8) defines a contraction mapping $T\hat{V}$ on the space in question, and so there exists a unique solution to $\hat{V} = T\hat{V}$.[7]

Given that it exists, it is evident from (8) and (9) that V is C^{n-1} with respect to m except at $m = m^*$. For $m > m^*$, $V_m = \phi$, since $q' = d' = 0$ in this range. For $m < m^*$,

$$V_m = \phi + \alpha\sigma \left[u'(q)\, \widehat{q}'(m) - \phi\right] = (1-\alpha\sigma)\phi + \alpha\sigma u'(q)\phi/z'(q), \tag{10}$$

Lemma 5 in the Appendix shows F is degenerate and $\phi_{+1} = \Phi(\phi)$ for some well behaved Φ in any equilibrium. Lemma 6 shows ϕ is bounded. Given this, the bargaining solution implies $v(m,s)$ is bounded and continuous. For the record, the term $v_0(s)$ is given by

$$v_0(s) = \alpha\sigma \int \{\phi d(\tilde{m}) - c[q(\tilde{m})]\}\, dF(\tilde{m}) + \alpha\delta[u(q^*) - c(q^*)] + U(X^*) - X^*.$$

[7] Operationally, the contraction generates the function $\hat{v}(m)$ and then we simply set $\hat{V}(m) = \hat{v}(m) + \phi m$. Note that this is not the method usually used to deal with unbounded returns (e.g., the method in Alvarez and Stokey [1998]).

since $\widehat{q}' = \phi/z'$ and $d' = 1$ in this range. Inserting z' explicitly, we have

$$V_m = (1-\alpha\sigma)\phi + \frac{\alpha\sigma\phi u'[\theta u' + (1-\theta)c']^2}{u'c'[\theta u' + (1-\theta)c'] + \theta(1-\theta)(u-c)(u'c'' - c'u'')}. \quad (11)$$

This implies that as $m \to m^*$ from below,

$$V_m \to (1-\alpha\sigma)\phi + \frac{\alpha\sigma\phi}{1+\theta(1-\theta)(u-c)(c''-u'')(u')^{-2}} < \phi. \quad (12)$$

Hence, the slope of V with respect to m jumps discretely as we cross m^*, which will important below.

The next thing to do is to check the concavity of V. To reduce notation, at this point we normalize $c(q) = q$ (with no loss of generality, since as discussed above this just means we are measuring special good output in utils). This reduces the algebra required to show that the derivative V_{mm} takes the same sign as $\Gamma + (1-\theta)\left[u'u''' - (u'')^2\right]$ for all $m < m^*$, where Γ is strictly negative but is otherwise of no concern. From this it is not possible to sign V_{mm} in general, due to the presence of u''', but it does give us some sufficient conditions for $V_{mm} < 0$. One such condition is $\theta \approx 1$. Another is $u'u''' \leq (u'')^2$, which follows if u' is log-concave (given our normalization). Hence, we have simple sufficient conditions to guarantee that V is strictly concave in m for all $m < m^*$, given any F and ϕ.[8]

To summarize the discussion to this point, we first described the value function in the decentralized market, $V(m, s)$, in terms of $W(m, s)$ and the terms of trade. We then derived some properties of the value function in the

[8]To understand the issues, observe that $V_{mm} = (q')^2 u'' + u'q''$ for all $m < m^*$. The first term is negative but the second takes the sign of q'', which may be positive. Intuitively, $q'' > 0$ means that having more money gets you a lot better deal in bargaining. The assumption $\theta = 1$ implies $q(m) = \phi m$ (given our normalization), and hence $V_{mm} < 0$ for sure. If $\theta < 1$, however, $q(m)$ is nonlinear and we need some condition such as log-concavity to restrict the degree of nonlinearity.

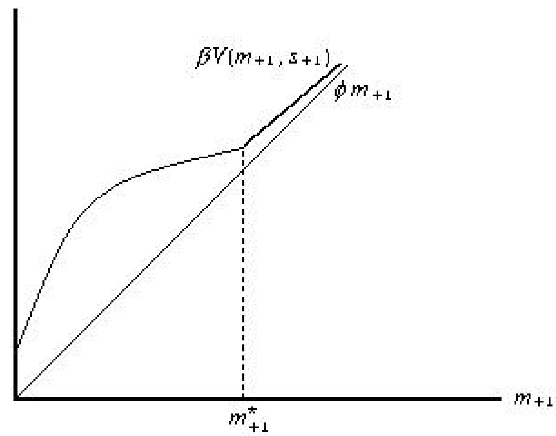

Figure 3: Value function.

centralized market, including $W(m, s) = W(0, s) + \phi m$. This made it easy to solve the bargaining problem for $q(m)$ and $d(m)$. This allowed us to simplify Bellman's equation considerably, to establish existence and uniqueness of a solution, and to give several properties of the V, including differentiability and under certain assumptions strict concavity in m for all $m < m^*$. The proof of existence and uniqueness for V was outlined assuming a steady state, but the Appendix proves it even if ϕ and F vary over time.[9]

Given these results we can now solve the problem of an agent deciding how much cash to take out of the centralized market: $\max_{m_{+1}}\{-\phi m_{+1} + \beta V(m_{+1}, s_{+1})\}$. First, Lemma 3 in the Appendix proves $\phi \geq \beta\phi_{+1}$ in any equilibrium by a simple arbitrage argument. This implies $-\phi m_{+1} + \beta V(m_{+1}, s_{+1})$ is nonincreasing for $m_{+1} > m^*_{+1}$. But recall from (12) that the slope of $V(m_{+1}, s_{+1})$ jumps discretely as m_{+1} crosses m^*_{+1}, as shown in Figure 3. From this it is apparent that any solution m_{+1} must be strictly less than

[9]This is important because we want to establish that under our assumptions *any* equilibrium (and not only any stationary equilibrium) has certain features.

m^*_{+1}. This is good to know since now the bargaining solution says $d = m$ and $q = \widehat{q}(m) < q^*$. Moreover, given that V is strictly concave for $m_{+1} < m^*_{+1}$, there exists a *unique* maximizer m_{+1}. This is especially good to know since then F is degenerate: $m_{+1} = M$ for all agents in any equilibrium.[10]

The first order condition for m_{+1} is

$$-\phi + \beta V_1(m_{+1}, s_{+1}) \leq 0, \quad = 0 \text{ if } m_{+1} > 0. \tag{13}$$

An *equilibrium* can now be defined as a value function $V(m, s)$ satisfying Bellman's equation, a solution to the bargaining problem given by $d = m$ and $q = \widehat{q}(m)$, and a bounded path for ϕ such that (13) holds at every date with $m = M$. Implicit in this definition is F, but it is degenerate. Of course, there is always a nonmonetary equilibrium where $\phi = 0$ at every date; in this case decentralized trade shuts down, although the centralized markets are still active. In what follows we focus on *monetary* equilibria, where $\phi > 0$ and (13) holds with equality.

2.3 Results

We now reduce the equilibrium conditions to one equation in one unknown. First insert V_m from (10) into (13) at equality to get

$$\phi = \beta\phi_{+1}\left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{+1})}{z'(q_{+1})}\right].$$

Then insert $\phi = z(q)/M$ from (6) to get

$$z(q) = \beta z(q_{+1})\left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{+1})}{z'(q_{+1})}\right]. \tag{14}$$

[10] Although these results are clear from the figure, they are proved rigorously in Lemmas 4 and 5 in the Appendix, where we note that we do note use the value function V (so that we can use the results in proving the general existence of V).

This is a simple difference equation in q. A monetary equilibrium can now be characterized as any path for q that stays in $(0, q^*)$ and satisfies (14).

Things simplify a lot in some cases. First, consider $\theta = 1$ (take-it-or-leave-it offers by buyers). In this case, (6) tells us $z(q) = c(q) = q$ (given our normalization) and then (14) reduces to

$$q = \beta q_{+1} \left[1 - \alpha\sigma + \alpha\sigma u'(q_{+1})\right].$$

Second, regardless of θ, if we restrict attention to steady states where $q_{+1} = q$, (14) becomes

$$1 = \beta \left[1 - \alpha\sigma + \alpha\sigma \frac{u'(q)}{z'(q)}\right].$$

For convenience, we rearrange this as

$$e(q) = 1 + \frac{1-\beta}{\alpha\sigma\beta}, \tag{15}$$

where $e(q) = u'(q)/z'(q)$. From now on we focus on steady states, and relegate dynamics to a companion paper (Lagos and Wright [2002]).

Consider first steady states with $\theta = 1$, which means $z(q) = q$ and (15) is

$$u'(q) = 1 + \frac{1-\beta}{\alpha\sigma\beta}. \tag{16}$$

Since $u'(q^*) < 1 + \frac{1-\beta}{\alpha\sigma\beta}$, a monetary steady state $q^s \in (0, q^*)$ exists iff $u'(0) > 1 + \frac{1-\beta}{\alpha\sigma\beta}$, and if it exists it is obviously unique. More generally, for any θ a monetary steady state exists if $e(0) > 1 + \frac{1-\beta}{\alpha\sigma\beta}$, but we cannot be sure of uniqueness since we do not know the sign of e'. However, we claim that if u' is log-concave then $e' < 0$ and q^s is unique. We also claim that $e(q)$ is increasing in θ; hence if it is unique then $\partial q^s/\partial\theta > 0$. It is also clear that $\partial q^s/\partial\alpha > 0$, $\partial q^s/\partial\sigma$ and $\partial q^s/\partial\beta > 0$. For $\theta = 1$, notice $q^s \to q^*$ as $\beta \to 1$;

for $\theta < 1$, however, q is bounded away from q^* even in the limit as $\beta \to 1$. We return to this below.[11]

We summarize the main findings in a Proposition. The proof follows directly from the discussion in the text, although two technical claims in the previous paragraph need to be established: that e is increasing in θ, and that e is decreasing in q for any θ if u' is log-concave. This is done in the Appendix.

Proposition 1 *Any monetary equilibrium implies that* $\forall t > 0$, $m = M$ *with probability* 1 *(*F *degenerate),* $d = m$, *and* $q = \widehat{q}(m) < q^*$. *Given any* $\theta > 0$, *a steady state* $q^s > 0$ *exists if* $e(0) > 1 + \frac{1-\beta}{\alpha\sigma\beta}$. *It is unique if* $\theta \approx 1$ *or* u' *is log-concave, in which case* q^s *is increasing in* β, α, σ *and* θ. *It converges to* q^* *as* $\beta \to 1$ *iff* $\theta = 1$.

We close this subsection by returning to the issue of nonnegativity. Recall that we have not imposed $Y \geq 0$, but we can now give conditions to rule out $Y < 0$ in equilibrium. To make the point clearly, assume the economy begins at $t = 0$ in the second subperiod, with the centralized market. Ignoring nonnegativity, we have shown $X = X^*$ and $m_{+1} = M$, and so an agent endowed with m supplies $Y(m) = X^* + \phi(M - m)$. From (6), ϕ is an increasing function of q and, since $q < q^*$, ϕ is bounded above by $\phi^* = [\theta c(q^*) + (1-\theta)u(q^*)]/M$. Hence, in the worse case scenario $\phi = \phi^*$, we have $Y(m) \geq 0$ if

$$m \leq M + \frac{X^*}{\phi^*} = M\left[1 + \frac{X^*}{\theta c(q^*) + (1-\theta)u(q^*)}\right]. \tag{17}$$

[11] In case it is not obvious, we also mention that the model displays classical neutrality: since M has vanished from (14), the set of equilibrium q paths is independent of M and all nominal variables are proportional to M. Real variables will not generally be independent of the growth rate of M, however, as we will see in the next section.

As long as (17) holds for all agents at $t = 0$, they all choose $Y_0 \geq 0$.

One can regard this as a restriction on the exogenous initial distribution of money F_0 – basically, it cannot be too disperse – or, for a given F_0, a restriction on preferences though X^* and q^*. For $t > 0$, to guarantee $Y_t \geq 0$ we need (17) to hold for every agent entering the centralized market with the endogenous m they brought in from decentralized trading. The binding agent is the one with the greatest m, which is anyone who sold goods on the decentralized market and now holds $2M$ dollars. Setting $m = 2M$ in (17) and simplifying, we get

$$X^* \geq \theta c(q^*) + (1 - \theta) u(q^*). \tag{18}$$

This guarantees that in equilibrium $Y_t \geq 0$ for all $t > 0$.

2.4 Discussion

Trejos and Wright (1995) discuss a model where agents can hold any $m \in \mathbb{R}_+$ and present a Bellman equation essentially identical to (1), except that since there are no centralized markets, $W(m) = \beta V(m)$. With no centralized meetings the distribution F is not degenerate, and very little can be done with the model. Hence, Trejos and Wright (1995) and also Shi (1995) studied the model under the restriction $m \in \{0, 1\}$. This keeps things tractable but is obviously a severe restriction. Molico (1999) allowed agents to hold any $m \in \mathbb{R}_+$ and studied the model numerically.[12] Although computational results can be useful, there is also something to be said for analytic tractability. For one thing, if one has to resort to computation it is hard to say much about

[12] Other models that relax the restriction $m \in \{0, 1\}$ to a greater or lesser extent include Green and Zhou (1997), Camera and Corbae (1999), Taber and Wallace (1999), Zhou (1999) and Berentsen (2002).

existence, uniqueness, and other general properties even for steady states, to say nothing of dynamics.

The analysis in this paper is much simpler due to the presence of the centralized market, which does several things. First, it yields the linearity of W with respect to m, which simplifies Bellman's equation and the bargaining solution considerably. Additionally, as all agents take the same m_{+1} out of the centralized market F is degenerate. This comes to us at a cost: we miss the way changes in parameters or policy variables might affect an endogenous distribution and how this could affect other variables, including welfare. But the advantage is that we can prove a lot of results in our framework, and it is very easy to put to use both qualitatively and quantitatively.

There is a related approach due to Shi (1997), where there is also a degenerate F but for a different reason.[13] His model assumes the fundamental decision-making unit is not an individual but a family with a continuum of agents. Each household's members search in a standard decentralized market, but at the end of each round they meet back at the homestead to share their money. By the law of large numbers, each family has the same total money, and it divides it evenly among its buyers for the next round. Hence, all buyers in the decentralized market have the same m. The large-household "trick" is a similar device to our assumption of a centralized market, at least in the sense that both render F degenerate.

While both approaches are useful, it seems incumbent upon us to suggest some relative merits for our "trick." First, some people view the infinite family structure as unappealing for a variety of reasons. Whether or not one

[13] See also Shi (1998,1999), Rauch (2000), Head and Shi (2000), Berentsen and Rocheteau (2000a, 2000b), Berentsen, Rocheteau and Shi (2001), Faig (2001), and Head and Kumar (2001).

agrees with this view, it seems good to have an alternative lest people think that tractable monetary models with search-theoretic foundations require infinite families. Second, there are some technical complications that arise in family models because infinitesimal agents bargain over trades that benefit larger decision-making units (Rauch [2000]; Berentsen and Rocheteau [2002]). This is not the case here since individuals bargain for themselves. Hence, we can use standard bargaining theory with impunity. Indeed, the linearity of $W(m)$ makes bargaining extremely simple here.

Third, there is the related but distinct point that individual incentive conditions are not taken into account in family models: agents act not in their own self interest, but in accordance with rules prescribed by the head of the household. Every time an agent produces to acquire cash he suffers a cost, but in principle he could report back to the clan without cash and claim he had no customers. This would save the cost with no implication for his future payoff. For the family structure to survive, then, agents must act in the interest of the household and not themselves. In our model, agents produce for money not out of brotherly love, but because they want cash for their own consumption.

Fourth, we simply find our model more transparent and easier to use – not least because (to paraphrase Kiyoyaki and Moore from the epigram) it relies on the basic tool of our trade, competitive markets. For many extensions and applications one may want to introduce centralized trading anyway, perhaps a centralized bond, capital, or labor market. In our model centralized markets are already up and running and we do not need to add one to the infinite-family structure. Still, having all this, we reiterate that large families and centralized markets are both potentially useful modeling devices, and the choice may sometimes come down to tastes or to the particular application at hand.[14]

Of course, we actually need more than centralized meetings to make F degenerate: we also need quasi-linear preferences. Given a general utility function for goods in the centralized market $\hat{U}(X, Y)$, agents will still try to adjust their money holdings by producing or consuming different quantities, but only if $\hat{U} = U(X) - Y$ or $\hat{U} = X - C(Y)$ will they necessarily all adjust to $m = M$. Our preferences are special, but while it is easy to set up the generalized version it does not entail much of a gain in tractability over a model with no centralized meetings. Hence, here we prefer to pursue a specification that, while special, is simple. It is an open question whether the wealth effects we ignore are empirically important. Presumably, they will be relatively important for some issues and not for others.

3 Applications and Extensions

3.1 Inflation

We begin this section by generalizing the model to allow the money supply to grow over time, say $M_{+1} = (1 + \tau)M$. New money is injected as a lump-sum transfer, or tax if $\tau < 0$, that occurs after agents leave the centralized

[14] A generalization of these models may be worth considering, since they are all based on somewhat special assumptions about the timing and nature of meetings. One can imagine a general pattern of meetings over time – some in small groups, some in centralized markets, some in familes. Pure bilateral matching and pure centralized markets are obviously special cases, and so is the setting here where we alternate between these two pure cases each period. We could in principle allow n periods of bilateral matching followed by a centralized market or family gathering (our case is $n = 1$; pure bilateral matching is $n = \infty$). Alternatively, we could allow some large groups to meet every alternate period but keep other individuals out of these meetings. With such generalizations F will not be degenerate, but it may have a fairly simple structure.

market. Bellman's equation becomes

$$V(m,\phi) = \max_{m_{+1}} \left\{ v(m,\phi) + \phi m - \phi m_{+1} + \beta V\left(m_{+1} + \tau M, \phi_{+1}\right) \right\}$$

where v is defined in (9) and we write $s = \phi$ since F will still be degenerate. In general, τ could vary with time, but if it is constant then it makes sense to consider steady states where q and real balances $z = \phi M$ are constant; that is, where $\phi_{+1} = \phi/(1+\tau)$. As in the previous section, $\phi \geq \beta\phi_{+1}$ is necessary for an equilibrium to exist, by Lemma 3. This implies $\tau \geq \beta - 1$.

Following the same procedure as before, we insert V_m and $\phi = z(q)/M$ into $\phi = \beta V_m$ to get the generalized version of (14):

$$\frac{z(q)}{M} = \beta \frac{z(q_{+1})}{M_{+1}} \left[1 - \alpha\sigma + \alpha\sigma \frac{u'(q_{+1})}{z'(q_{+1})} \right]. \tag{19}$$

Given $M_{+1} = (1+\tau)M$, with τ constant, if we focus on steady states things simplify a lot. After some algebra, the generalized version of (15) is

$$e(q) = 1 + \frac{1 - \beta + \tau}{\alpha\sigma\beta} \tag{20}$$

where again $e(q) = u'(q)/z'(q)$. Assuming a unique monetary steady state q^s exists, which it will under the same conditions given in Proposition 1, $\partial q^s/\partial\tau < 0$.

From (20) it appears that all one needs to do to achieve the efficient outcome $q^s = q^*$ as a steady state monetary equilibrium is to set $\tau = \tau^* = \beta - 1 - \alpha\sigma\beta\left[1 - e(q^*)\right]$. However, as we said above, the simple arbitrage argument in Lemma 3 implies $\phi \geq \beta\phi_{+1}$ and this implies $\tau \geq \beta - 1$. Hence, there is a bound on feasible policies: we cannot contract the money supply any faster than Friedman's (1969) rule, which is to deflate at the rate of time preference, $\tau^F = \beta - 1$. Any attempt to contract the money supply faster than this and the monetary equilibrium will break down.

If $\theta = 1$ and we set $\tau = \tau^F$ then from (20) the steady state is indeed $q^s = q^*$. If $\theta < 1$, however, then at $\tau = \tau^F$ we have $q^s < q^*$. The Friedman rule always maximizes q^s, which is the optimal constrained policy, but it achieves the efficient outcome q^* iff $\theta = 1$. The reason is that in the model there are two types of inefficiencies, one due to β and one to θ. To describe the first effect, note that when you accept cash you get a claim to future consumption, and because $\beta < 1$ you are willing to produce less for cash than the q^* you would produce if you could turn it into immediate consumption (recall $q^s \to q^*$ as $\beta \to 1$ when $\theta = 1$). The Friedman rule simply generates a rate of return on money due to deflation that compensates for discounting.

The wedge due to $\beta < 1$ is standard, and the only difference from, say, a cash-in-advance model on this dimension is that here the frictions show up explicitly: (20) makes it clear that for a given β and τ the inefficiency gets worse as $\alpha\sigma$ gets smaller. The novel effect here is the wedge due to $\theta < 1$. One intuition for this is the notion of a *hold-up problem*. Think of an agent who carries a dollar into next period as making an investment with cost ϕ. When he spends the money he reaps all of the returns to his investment iff $\theta = 1$; otherwise the seller "steals" part of the surplus. Thus $\theta < 1$ reduces the incentive to invest, which lowers the demand for money and hence q. Therefore $\theta < 1$ implies $q^s < q^*$ even at the Friedman rule.[15]

The wedge due to $\theta < 1$ does not come up in the usual reduced-form models, but it can be important. Consider the welfare cost of inflation. When $\theta = 1$, welfare, as measured by the payoff of the representative agent

[15] Recall Hosios' (1990) general condition for efficiency in search models, which bascially says the bargaining outcome should split the gains from trade so as to compensate each party for his contribution to the match-specific surplus. Here the match-specific surplus in a single-coincidence meeting is all due to the buyer, since the bargaining solution depends on his money holdings but not those of the seller. Hence, efficiency requires $\theta = 1$.

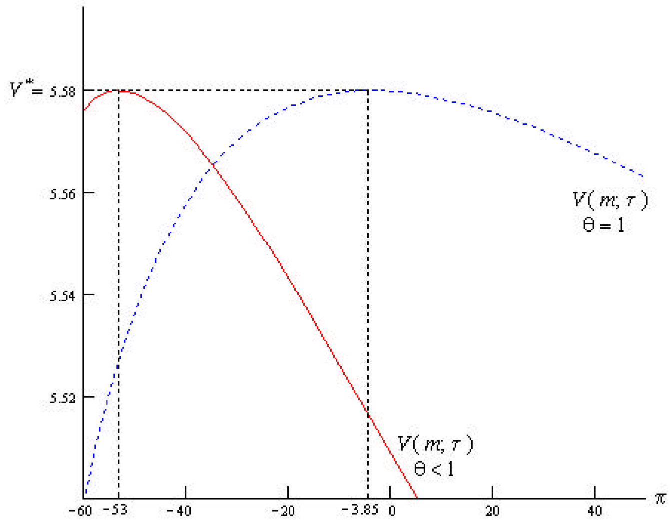

Figure 4: Welfare effects of inflation.

V, is maximized at the Friedman Rule τ^F and achieves the efficient solution

$$V^* = \frac{\alpha(\delta + \sigma)\left[u(q^*) - c(q^*)\right] + U(X^*) - X^*}{1 - \beta}.$$

See Figure 4 (drawn for the calibrated parameter values discussed below). With $\theta = 1$, just like in a typical reduced-form model, small deviations from $\tau = \beta - 1$ have very small effects on welfare by the Envelope Theorem. When $\theta < 1$, $\tau = \tau^* < \tau^F$ would achieve V^* if it were feasible, but it is not. At the constrained optimum the slope of V with respect to τ is steep, so a moderate inflation will have a much bigger welfare cost.

3.2 Calibration

Here we calibrate the model to quantify the effects identified above. While we do not intend this to be the last word on calibrating this model, it seems important to illustrate these effects can be sizable, and also that the model can easily be taken to the data. We use the utility function

$$u(q) = \frac{(b+q)^{1-\eta} - b^{1-\eta}}{1-\eta}, \tag{21}$$

where $\eta > 0$ and $b \in (0,1)$. This generalizes the standard constant relative risk aversion preferences by allowing $b \neq 0$, which forces $u(0) = 0$. It implies relative risk aversion is given by $\eta q/(b+q)$, which is increasing in q, while absolute risk aversion is given by $\eta/(b+q)$, which is decreasing in q. We normalize $c(q) = q$ so the efficient solution is $q^* = 1 - b$.

Our base parameter values are chosen as follows. First, we set the period to one week (unlike in some models, we can easily calibrate to any frequency). This means setting $\beta = (1+r)^{-1/52}$ where r is an estimate of the annual real interest rate, which we take to be 4%. We choose the base value of τ to generate an annual inflation rate of 4%. In terms of the arrival rates, we normalize $\alpha = 1$ since it is only the products $\alpha\delta$ and $\alpha\sigma$ that matter. We set $\delta = 0$ since direct barter as relatively rare in modern economies, but this does not matter at all for the results. We then set σ to match velocity. If we take M to be the monetary base, the weekly velocity with respect to consumption (nondurables and services) was on average 0.2 for the period 1980-2000. This implies $\sigma = 0.1$.[16]

The model predicts all sales in the decentralized market have a markup (price over marginal cost) equal to $\mu = \phi M/q$. Numerically, this markup depends on θ as well as the preference parameters b and η. We calibrate θ and b so that in the benchmark steady state equilibrium the model generates a markup of $\mu = 1.17$ (as in Rotemberg and Woodford [1995]) and relative

[16] It would be relatively easy to endogenize α through a search intensity decision or σ through a specialization decision, and thereby have velocity respond to policy or other shocks; we leave this for future work.

risk aversion for consumption equal to 1 (to facilitate comparison with some well-known inflation studies such as Cooley and Hansen [1989]). This leaves η as a free parameter; we take as a benchmark $\eta = 5$, but we tried other values and, as we report below, the results were not very sensitive to this choice. Table 1 summarizes these parameter values.

α	δ	σ	θ	η	b	β	τ
1.0	0.0	0.1	0.828	5.0	0.773	$(1.04)^{-1/52}$	$(1.04)^{1/52}$

Table 1: Baseline calibration.

It is easy to solve (20) numerically for the steady state as a function of τ, $q^s = q^s(\tau)$. We then compute the welfare cost of inflation two ways. First, we simply consider the percent reduction in q from the value that would arise under the Friedman rule: $w_1 = \left[q^s(\tau^F) - q^s(\tau)\right]/q^s(\tau)$. This captures how much inflation affects the value of money, or the amount of special goods one can buy with one's money. However, this measure does not capture the economic intuition behind Figure 4, since it does not incorporate the idea that the utility function can be steep at the constrained optimal policy τ^F. Hence, we also ask how much consumption agents would be willing to give up to change policy from τ to the Friedman rule.

To be more precise, notice that in equilibrium for any τ we have

$$(1-\beta)V = U(x^*) - x^* + \alpha\sigma\left\{u[q^s(\tau)] - q^s(\tau)\right\}. \qquad (22)$$

Suppose we run the Friedman rule but reduce consumption and production of general goods in the centralized market by a factor w_2; then

$$(1-\beta)V = U[x^*(1-w_2)] - x^*(1-w_2) + \alpha\sigma\left\{u[q^s(\tau^F)] - q^s(\tau^F)\right\}. \qquad (23)$$

Our second welfare measure is the value of w_2 that equates the right hand sides of (22) and (23). To calculate this we use $U(x) = u(x)$ – i.e. the same functional form for general and special goods. The measure w_2 captures the idea that agents are willing to give up a bigger fraction of consumption when they are close to than when they are far from the efficient level.

Figure 5 shows the welfare measures w_1 and w_2 as functions of the annual inflation rate π, where π ranges from the Friedman rule to 50%. It also shows the corresponding measures $\tilde{w}_1$ and $\tilde{w}_2$ that one would get if one ignored the holdup problem by setting $\mu = 1$.[17] Notice $\tilde{w}_i < w_i$, for $i = 1,2$, but the difference is much bigger for the w_2 measures. A key observation here is that not only do we have $w_2 > \tilde{w}_2$, but also w_2 is steep near the Friedman rule, due to the envelope theorem as discussed above. Hence, w_2 will be especially large relative to $\tilde{w}_2$ at fairly low to moderate inflation rates.

Table 2 reports the actual numbers. Notice that even a constant price policy $\pi = 0$ implies a sizable drop in q of $w_1 = 0.79\%$ from the q implied by the Friedman rule. This is equivalent to a reduction in general good production and consumption of $w_2 = 1.44\%$. The corresponding numbers when we ignore the holdup problem by setting $\mu = 1$ are $\tilde{w}1 = 0.75\%$ and $\tilde{w}_2 = 0.23\%$. When we get as high as 10% inflation we find that the drop in q is $w_1 = 2.72\%$ and this is equivalent to a reduction in general good consumption of $w_2 = 2.73\%$. The corresponding numbers when we ignore the holdup problem are $\tilde{w}_1 = 2.58\%$ and $\tilde{w}_2 = 0.80\%$. The general conclusion is clear: the distortion due to $\theta < 1$ makes a big difference, especially at relatively low inflation rates and especially for the measure w_2.

[17] To obtain $\tilde{w}_i$, the model was recalibrated so that μ and relative risk aversion are both 1 at the benchmark steady state. The implied parameter values are $\theta = 1$ and $b = .798$.

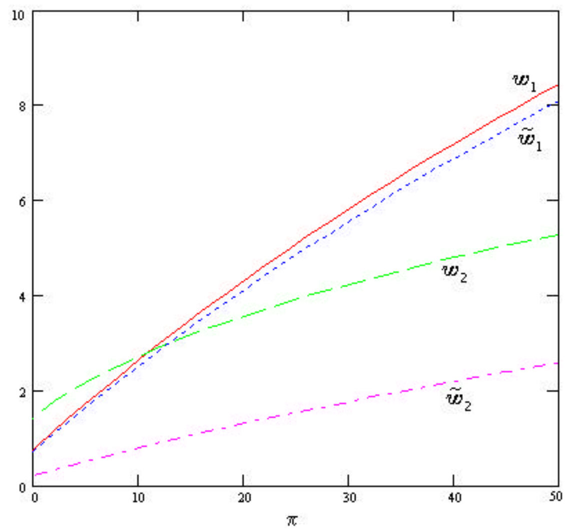

Figure 5: Welfare cost for baseline calibration.

To put this into perspective, consider the welfare costs of inflation Lucas (2000) calculates in a reduced form model. He finds that a drop in inflation from 10% to 0% is worth slightly less than 1% of income (the exact number depending on some details). Here, a drop in inflation from 10% to 0% is worth 2.32% of general goods consumption. If we do the same calculation ignoring the holdup problem by setting $\theta = 1$, a 10% inflation is worth only 0.76% of general goods consumption, consistent with the Lucas estimate. Other studies find results about the same as those in Lucas, so we conclude that our results are due mainly to the distortion due to the bargaining wedge.[18]

[18] In Cooley and Hansen's (1989) cash-in-advance model the welfare cost of a 10% inflation relative to the Friedman rule is 0.152% of consumption when the cash-in-advance constraint is monthly, and 0.52% if the constraint is quarterly. In Cooley and Hansen's (1991) model with cash and credit goods, the result is 0.27%, and by adding distorting capital and labor taxes they get it up to 0.68%. Gomme (1993) finds even lower numbers in an endogenous growth model. In a cash-in-advance model, Wu and Zhang (2000) argue that the welfare cost of inflation is larger if one introduces monopolistic competition, and also give some additional references.

π (% annual inflation)	w_2	$\tilde{w}_2$	w_1	$\tilde{w}_1$
-3.85	0	0	0	0
0	1.44	0.23	0.79	0.75
5	2.18	0.52	1.78	1.68
10	2.73	0.80	2.72	2.58
15	3.18	1.06	3.63	3.45
20	3.57	1.31	4.50	4.29
25	3.92	1.54	5.35	5.10
30	4.23	1.77	6.17	5.88
35	4.52	1.99	6.96	6.64
40	4.79	2.19	7.73	7.37
45	5.05	2.39	8.47	8.09
50	5.28	2.58	9.20	8.78

Table 2: The welfare cost of inflation.

One would like to know how robust our results are to alternative calibrations. In terms of the free parameter η, we tried values of 0.5, 10 and 20 and the results were not very different: at $\pi = 10\%$, the results are $w_2 = 2.78\%$, 2.70% and 2.69% (for these alternative values of η we recalibrated b and θ to keep μ at 1.17 and risk aversion at 1). We also considered changing the target level of μ and risk aversion. Table 3 reports w_2 for a 10% inflation, where in each case we kept $\eta = 5$ and recalibrated b and θ to generate each (μ, RRA) pair. Naturally, w_2 is larger when we assume a bigger markup μ, since this exacerbates the holdup problem. Also, reducing risk aversion increases w_2, since this makes q more sensitive to π. But the basic result remains that we are finding sizable welfare effects.

As an additional robustness check, we considered the following alternative calibration procedure. Rather than setting θ to match the markup μ, we tried setting $\theta = 1/2$ in the interest of symmetry. Given this, we then chose b to match the markup. This leaves η, which we set to $\eta = 5$ (the same

$\mu \setminus RRA$	0.5	1.0	1.5	2.0
1.0	1.58	0.80	0.53	0.40
1.17	4.75	2.73	2.09	1.75
1.3	6.37	3.30	2.51	2.10
1.4	7.49	3.65	2.73	2.29

Table 3: Sensitivity analysis.

benchmark as in the previous procedure). Calibrating to $\mu = 1.17$, we find that $w_2 = 2.97\%$ for 0% inflation; while $w_2 = 6.01\%$ for 10% inflation. These numbers are bigger than in our base case, but mainly due to the fact that the implied risk aversion is low for these parameters, which as we said above makes q more sensitive to π. Hence, we conclude that while the exact results depend somewhat on the exact parameterization, the essential finding is that the model can generate large welfare effects from inflation.

3.3 Uncertainty

So far we have dealt with deterministic environments, and shown that the constraint $d \leq m$ is always binding. We now show that with stochastic shocks the constraint does not have to bind with probability 1, which could be important in some applications since it makes velocity vary. We begin with match-specific uncertainty: when two agents meet they draw $\varepsilon = (\varepsilon_b, \varepsilon_s)$ from $H(\varepsilon)$, independently across meetings, implying utility and production cost in that meeting are $\varepsilon_b u(q)$ and $\varepsilon_s q$. For simplicity, we set $c(q) = q$, $\theta = 1$, and $\delta = 0$ in this section, mainly to reduce the notation.

The bargaining solution generalizes (5),

$$q = \begin{cases} \widehat{q}(m, \varepsilon) & \text{if } m < m^*(\varepsilon) \\ q^*(\varepsilon) & \text{if } m \geq m^*(\varepsilon) \end{cases} \quad \text{and } d = \begin{cases} m & \text{if } m < m^*(\varepsilon) \\ m^*(\varepsilon) & \text{if } m \geq m^*(\varepsilon) \end{cases}$$

where $\widehat{q}(m, \varepsilon) = \phi m / \varepsilon_s$, $u'[q^*(\varepsilon)] = \varepsilon_s / \varepsilon_b$, and $m^*(\varepsilon) = \varepsilon_s q^*(\varepsilon) / \phi$ (these are simple because $\theta = 1$ here). For a given ε_s, buyers with high realizations of ε_b will spend all their cash but those with low ε_b may not. Let $C = \{\varepsilon | q^*(\varepsilon) > \phi m/\varepsilon_s\}$ be the set of realizations such that $d = m$. Bellman's equation is still given by (8) but now

$$v(m,s) = \alpha\sigma \int \{\varepsilon_b u[q(m,\varepsilon)] - \phi d(m,\varepsilon)\} \, dH(\varepsilon) + U(X^*) - X^*$$

Since $v_{mm} < 0$, V is strictly concave and F is degenerate, as in the deterministic case. Substituting V_m into the first order condition $\phi = \beta V_m$, after rearranging we get

$$\int_C \left[\left(\frac{\varepsilon_b}{\varepsilon_s} \right) u' \left(\frac{M\phi_{+1}}{\varepsilon_s} \right) - 1 \right] dH(\varepsilon) = \frac{\phi - \beta\phi_{+1}}{\alpha\sigma\beta\phi_{+1}}. \tag{24}$$

If $u'(0)$ is big, there is a unique monetary steady state $\phi^s > 0$, and from this $q = q(M,\varepsilon)$ and $d = d(M,\varepsilon)$ are obtained from the bargaining solution. Clearly $d \leq m$ must bind with positive probability, since otherwise (24) could not hold. However, it is an easy exercise to work out examples where it binds with probability less than 1.[19]

We now return to $\varepsilon_b = \varepsilon_s = 1$ and consider uncertainty in M. We first consider random transfers across agents: before the start of trade, an agent

[19] Rather than interpreting ε as match-specific, the same results hold if it is an i.i.d. aggregate shock. Now suppose ε is an aggregate shock with conditional distribution $H(\varepsilon_{+1}|\varepsilon)$. Then Bellman's equation satisfies a version of (8) where

$$v(m,\varepsilon) = \alpha\sigma \{\varepsilon_b u[q(m,\varepsilon)] - \phi d(m,\varepsilon)\} + U(X^*) - X^*.$$

Substituting V_m into $\phi = \beta V_m$, we get

$$\phi(\varepsilon) = \beta \int \phi(\varepsilon_{+1}) \left(1 + I(\varepsilon_{+1}) \alpha\sigma \left\{ \frac{\varepsilon_{+1,b}}{\varepsilon_{+1,s}} u' \left[\frac{\phi(\varepsilon_{+1}) M}{\varepsilon_{+1,s}} \right] - 1 \right\} \right) dH(\varepsilon_{+1}|\varepsilon)$$

where $I(\varepsilon)$ is an indicator function that equals 1 if $\varepsilon \in C$ and 0 otherwise. In general this is a functional equation in $\phi(\cdot)$. In the i.i.d. case the right hand side is independent of ε, so $\phi(\varepsilon)$ is constant and things are the same as the match-specific case.

who brougt m to the decentralized market gets $m+\rho$ dollars, where ρ has distribution $H(\rho)$ with $E\rho = 0$. Bellman's equation is still given by (8), but now

$$v\left(m,s\right)=U\left(X^{*}\right)-X^{*}+\alpha\sigma\int\left\{u[q\left(m+\rho\right)]-\phi d\left(m+\rho\right)\right\}dH\left(\rho\right).$$

Again, $v_{mm} < 0$ and F is degenerate. Substituting V_m into $\phi = \beta V_m$ now yields

$$\int_{\underline{\rho}}^{\hat{\rho}}\left\{u'[(M+\rho)\,\phi_{+1}]-1\right\}dH(\rho)=\frac{\phi-\beta\phi_{+1}}{\alpha\sigma\beta\phi_{+1}}. \tag{25}$$

where $\hat{\rho} = q^*/\phi_{+1} - M$ is the minimum transfer that makes the constraint slack. If $u'(0)$ is big there exists a unique monetary steady state ϕ^s.

To analyze risk, consider a family of distributions $H\left(\rho,\Sigma\right)$ where $\Sigma_2 > \Sigma_1$ implies $H\left(\rho,\Sigma_2\right)$ is a mean preserving spread of $H\left(\rho,\Sigma_1\right)$; that is, $\Xi\left(\tilde{\rho},\Sigma\right) = \int_{\underline{\rho}}^{\widetilde{\rho}} H_2\left(\rho,\Sigma\right)d\rho \geq 0$ for any $\widetilde{\rho}$ with equality at $\widetilde{\rho} = \bar{\rho}$. Notice $H_2(\underline{\rho},\Sigma) = H_2\left(\overline{\rho},\Sigma\right) = 0$. Then $\partial\phi^s/\partial\Sigma$ is equal in sign to[20]

$$\begin{aligned}\Psi &= \int_{\underline{\rho}}^{\hat{\rho}(\phi)}\left\{u'[(M+\rho)\,\phi]-1\right\}dH_2(\rho,\Sigma)\\ &= \left[\left\{u'[(M+\rho)\,\phi]-1\right\}H_2(\rho,\Sigma)\right]_{\underline{\rho}}^{\hat{\rho}}-\int_{\underline{\rho}}^{\hat{\rho}(\phi)}\phi u''[(M+\rho)\,\phi]H_2(\rho,\Sigma)d\rho\\ &= -\phi u''\left(q^{*}\right)\Xi\left(\hat{\rho},\Sigma\right)+\int_{\underline{\rho}}^{\hat{\rho}(\phi)}\phi^{2}u'''[(M+\rho)\,\phi]\Xi\left(\rho,\Sigma\right)d\rho.\end{aligned}$$

[20] The second line follows if one integrates by parts. The last line follows as soon as one notices that the first term in the second line vanishes, because $u' = 1$ at $\hat{\rho}$ and $H_2(\underline{\rho},\Sigma) = 0$, and then integrates by parts again.

The first term is positive but the second depends on u'''; as long as $u''' \geq 0$, we have $\Psi > 0$ and more risk increases the value of money.

The effect of Σ on ϕ^s is due to a precautionary demand for money: given $u''' \geq 0$ an increase in risk makes agents want to hold more cash, which raises its value. However, it can be shown that an increase in risk unambiguously reduces welfare. This contrasts with some other models, where the distribution of money holdings F is nondegenerate in equilibrium, and random transfers may be welfare improving (Molico [1999]; Berentsen [2002]). The reason is that in those models random transfers can make the distribution of real balances less unequal. Here, the distribution of real balances is degenerate in equilibrium, so random transfers cannot help.

For our final experiment we let the growth rate of M be random with distribution $H(\tau_{+1}|\tau)$. Bellman's equation now satisfies a version of (8) with

$$v(m,\tau) = U(X^*) - X^* + \alpha\sigma \{u[q(m)] - \phi d(m)\}.$$

Again F is degenerate, and the usual procedure yields

$$\phi = \beta \int_{C^c} \phi_{+1} dH(\tau_{+1}|\tau) + \beta \int_{C} \phi_{+1} \left[\alpha\sigma u'(\phi_{+1} m_{+1}) + 1 - \alpha\sigma\right] dH(\tau_{+1}|\tau)$$

where $C = \{\tau | (m + \tau M)\phi < q^*\}$. If we focus on stationary equilibrium, we can write

$$z(\tau) = \beta \int_{C^c} \frac{z(\tau_{+1}) dH(\tau_{+1}|\tau)}{1+\tau_{+1}} + \beta \int_{C} \frac{\{\alpha\sigma u'[z(\tau_{+1})] + 1 - \alpha\sigma\} z(\tau_{+1}) dH(\tau_{+1}|\tau)}{1+\tau_{+1}} \tag{26}$$

where $z(\tau)$ is real balances in state τ.

This is a functional equation in $z(\cdot)$. If shocks are i.i.d., $z(\tau)$ is constant and the constraint binds with probability 1; in this case z solves

$$u'(z) = 1 + \frac{\zeta^{-1} - \beta}{\beta\alpha\sigma}$$

where $\zeta = \int (1+\tau)^{-1} dH(\tau)$. When τ is persistent the inflation forecast depends on τ and so does z. Suppose $\tau \in \{\tau_1, \tau_2\}$, with $\tau_1 > \tau_2$, $pr(\tau = \tau_i | \tau_i) = p_i$, $pr(\tau = \tau_1 | \tau_2) = s_2$ and $pr(\tau = \tau_2 | \tau_1) = s_1$ where $p_1 > s_2$ (persistence). We write (26) as two equations in (z_1, z_2), and look for a solution (z_1^*, z_2^*) such that $z_i^* < q^*$. These equations, shown in Figure 6, can be rearranged as

$$\begin{aligned} z_1 &= z_1(z_2) = \left[\tfrac{p_1}{s_2} - \tfrac{\beta(1-\alpha\sigma)(p_1 p_2 - s_1 s_2)}{s_2(1+\tau_2)}\right] z_2 - \tfrac{\beta\alpha\sigma(p_1 p_2 - s_1 s_2)}{s_2(1+\tau_2)} u'(z_2) z_2 \\ z_2 &= z_2(z_1) = \left[\tfrac{p_2}{s_1} - \tfrac{\beta(1-\alpha\sigma)(p_1 p_2 - s_1 s_2)}{s_1(1+\tau_1)}\right] z_1 - \tfrac{\beta\alpha\sigma(p_1 p_2 - s_1 s_2)}{s_1(1+\tau_1)} u'(z_1) z_1. \end{aligned}$$

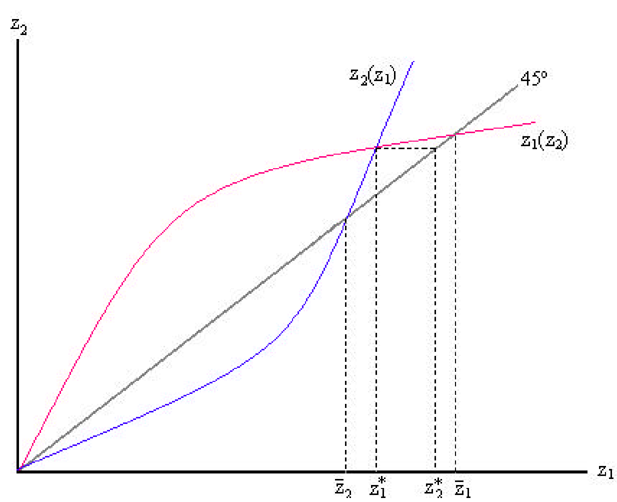

Figure 6: Equilibrium with random τ.

Notice $z_i(0) = 0$ and $\lim_{z\to\infty} z_i(z) = \infty$. It may be shown that $z_i' > 0$ as long as $-u''(z)z/u'(z) \geq 1$. Let $\overline{z}_i = z_i(\overline{z}_i)$ be the point where the $z_i(\cdot)$

function crosses the 45^o line (see the Figure), given by the solutions to

$$\begin{aligned}1+\alpha\sigma\left[u'\left(\overline{z}_2\right)-1\right] &= \frac{\left(1+\tau_1\right)\left(p_2-s_1\right)}{\beta\left(p_1p_2-s_1s_2\right)} \\ 1+\alpha\sigma\left[u'\left(\overline{z}_1\right)-1\right] &= \frac{\left(1+\tau_2\right)\left(p_2-s_1\right)}{\beta\left(p_1p_2-s_1s_2\right)}.\end{aligned}$$

These imply $\overline{z}_2 < \overline{z}_1$ if $\tau_1 > \tau_2$, and given $z_i' > 0$ this implies $z_1^* < z_2^*$. Hence, when the shocks to the money supply are persistent real balances are smaller in periods of high inflation, simply because in these periods beliefs about future inflation are higher.[21] While it may be interesting to purse these models with uncertainty, qualitatively or quantitatively, we leave this to future work.

4 Conclusion

This paper has presented a framework based on the explicit frictions that make money essential in the search-theoretic literature, but without the extreme restrictions usually made in those models about individuals' money holdings. The key innovation is that agents sometimes interact in decentralized meetings and sometimes in a centralized market. Although this is obviously not the most general case, we concentrated on the situation where these two types of meetings occur one right after the other. In this case, if we assume agents have quasi-linear preferences over the good traded in the centralized market, the distribution of money holdings will be degenerate in equilibrium. This makes the model very tractable.

We characterized equilibria and showed how to use the model to study

[21] One detail remains: recall that the equilibrium was constructed conjecturing $z_i^* < q^*$. Since $z_i^* < \overline{z}_1$, for the conjecture to be correct it is sufficient to ensure $\overline{z}_1 \leq q^*$, which holds iff $\left(1+\tau_2\right)\left(p_1-s_2\right) \geq \beta\left(p_1p_2-s_1s_2\right)$.

some policy issues. The model displays classical neutrality, although inflation matters. The Friedman rule is optimal, although for values of the bargaining power parameter below 1 this policy does not achieve the first best. This has implications for the welfare cost of inflation, as we discussed in detail in a calibrated example. We also sketched how to extend the basic model to allow uncertainty in real and monetary variables. We think all of this constitutes progress in terms of bringing micro and macro economic models of monetary economics closer together. We also think that we have only scratched the surface, and much more could be done with the framework in terms of applications and extensions.

A Appendix

In this Appendix we first verify that the bargaining solutions are as claimed in the text. We then use these results to derive, without using value function V, certain properties any equilibrium must satisfy. We then use these properties to establish the existence and uniqueness of V. Finally, we provide some details for the proof of Proposition 1.

Lemma 1 *In a double coincidence meeting each agent produces q^* and no money changes hands.*

Proof. The symmetric Nash problem is

$$\max_{q_1, q_2, \Delta} \left[u(q_1) - c(q_2) - \phi\Delta\right]\left[u(q_2) - c(q_1) + \phi\Delta\right]$$

subject to $-m_2 \leq \Delta \leq m_1$, where q_1 and q_2 denote the quantities consumed by agents 1 and 2 and Δ is the amount of money 1 pays 2. There is a unique solution, characterized by the first order conditions

$$\begin{aligned} u'(q_1)\left[u(q_2) - c(q_1) + \phi\Delta\right] &= c'(q_1)\left[u(q_1) - c(q_2) - \phi\Delta\right] \\ c'(q_2)\left[u(q_2) - c(q_1) + \phi\Delta\right] &= u'(q_2)\left[u(q_1) - c(q_2) - \phi\Delta\right] \\ u(q_1) - u(q_2) + c(q_1) - c(q_2) - 2\phi\Delta &= \frac{(2/\phi)(\lambda_1 - \lambda_2)}{\{[u(q_1) - c(q_2) - \phi\Delta][u(q_2) - c(q_1) + \phi\Delta]\}^{-1/2}} \end{aligned}$$

where λ_i is the multiplier on agent i's cash constraint. It is easy to see that $q_1 = q_2 = q^*$ and $\Delta = \lambda_1 = \lambda_2 = 0$ solves these conditions. ■

Lemma 2 *In a single coincidence meeting the bargaining solution is given by (5).*

Proof. The necessary and sufficient conditions for (4) are

$$\begin{aligned} \theta\left[\phi d-c\left(q\right)\right]u'\left(q\right) &= \left(1-\theta\right)\left[u\left(q\right)-\phi d\right]c'\left(q\right) & (27)\\ \theta\left[\phi d-c\left(q\right)\right]\phi &= \left(1-\theta\right)\left[u\left(q\right)-\phi d\right]\phi & (28)\\ & \quad -\lambda\left[u\left(q\right)-\phi d\right]^{1-\theta}\left[\phi d-c\left(q\right)\right]^{\theta} \end{aligned}$$

where λ is the Lagrange multiplier on $d \leq m$. There are two possible cases: If the constraint does not bind, then $\lambda = 0$, $q = q^*$ and $d = m^*$. If the constraint binds then q is given by (27) with $d = m$, which is (6). ■

We now present some arbitrage-style arguments to establish that any equilibrium must satisfy certain conditions. These arguments do not use any properties of V or F.

Lemma 3 *In any equilibrium, $\beta\phi_{t+1} \leq \phi_t$ for all t.*

Proof. First, note that lifetime utility is finite in any equilibrium. Now suppose by way of contradiction that $\beta\phi_{t+1} > \phi_t$ at some t. In this case, an agent could raise his production of Y_t by dY and sell it for dY/ϕ dollars, then use the money at $t+1$ to reduce Y_{t+1} by dY/ϕ_{t+1} without changing anything else in his lifetime. Since utility is linear in Y, the net gain from this is $dY(-1+\beta\phi_{t+1}/\phi_t) > 0$. Hence $\beta\phi_{t+1} > \phi_t$ cannot hold in equilibrium. ■

Lemma 4 *In any equilibrium, $m_{t+1} < m^*_{t+1}$ for all t and for all agents.*

Proof. Suppose $m_{t+1} > m^*_{t+1}$ for some t and some agent. At t he can change Y_t by $dY < 0$ and carry $dm_{t+1} = dY/\phi_t$ fewer dollars into $t+1$. Given $m_{t+1} > m^*_{t+1}$, for small dY, the bargaining solution says this does not affect his payoff in the decentralized market. Hence, he can increase Y_{t+1} by

$dY_{t+1} = -dY\phi_{t+1}/\phi_t$ and not change anything else in his lifetime, for a net utility gain of $dY(1-\beta\phi_{t+1}/\phi_t) > 0$ by Lemma 3. This proves $m_{t+1} \leq m^*_{t+1}$.

To establish the strict inequality, assume $m_{t+1} = m^*_{t+1}$. Again change Y_t by $dY < 0$ and carry $dm_{t+1} = dY/\phi_t$ fewer dollars into $t+1$. If he buys in the decentralized market next period he gets a smaller q but the continuation value is the same from then on (he still spends all his money). If he is not a buyer then he can increase Y_{t+1} by $dY_{t+1} = -dY\phi_{t+1}/\phi_t$ and not change anything else in his lifetime. The net expected utility gain from this is

$$\begin{aligned} D &= -dY + \beta\left[\alpha\sigma u'(q_{t+1})\widehat{q}'(m_{t+1}) + (1-\alpha\sigma)\phi_{t+1}\right] dY/\phi_t \\ &= -dY(\phi_t - \beta\phi_{t+1})/\phi_t + \beta\alpha\sigma\left[u'(q_{t+1})\widehat{q}'(m_{t+1}) - \phi_{t+1}\right] dY/\phi_t. \end{aligned}$$

The first term on the right hand side is positive by Lemma 3, and the second is positive for small dY because then m_{t+1} is near m^*_{t+1} and this implies $\widehat{q}'(m_{t+1}) < \phi_{t+1}/u'(q_{t+1})$ from (7). ■

Lemma 5 *If $\theta \approx 1$ or u' is log concave then F is degenerate in any monetary equilibrium: all agents have $m_{t+1} = M$. Given F is degenerate, $\phi_t = G\left(\phi_{t+1}\right)$ for all t where G is a time-invariant continuous function.*

Proof. Consider the following sequence problem: given any path $\{\phi_t, F_t\}$ and m_0,

$$\max_{\{m_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[v\left(m_t, \phi_t, F_t\right) + \phi_t\left(m_t - m_{t+1}\right)\right]$$

where v is defined in (9) (which does not use V and is defined in terms date t variables only). We know v is C^{n-1}; thus, if a solution exists it satisfies the necessary conditions

$$\beta v_1\left(m_{t+1}, \phi_{t+1}, F_{t+1}\right) + \beta\phi_{t+1} - \phi_t \leq 0, \ = 0 \text{ if } m_{t+1} > 0. \tag{29}$$

We have $v_1(m, \phi, F) = \alpha\sigma\left[u'(q)\widehat{q}'(m) - \phi\right]$, since we know $m < m^*$, where $\widehat{q}'(m)$ is given in (7).

In any (monetary) equilibrium, at least one agent must choose $m_{t+1} > 0$, and for this agent

$$\beta v_1\left(m_{t+1}, \phi_{t+1}, F_{t+1}\right) + \beta\phi_{t+1} - \phi_t = 0. \tag{30}$$

A quick calculation verifies that if $\theta \approx 1$ or u' is log concave then $v_{11} < 0$, which implies (30) has a unique solution: all agents choose the same $m_{t+1} = M$. Hence F_{t+1} is degenerate in any monetary equilibrium. Finally, (30) implies $\phi_t = G\left(\phi_{t+1}\right)$, where G is continuous because v_1 is. ■

We have established F degenerate in any equilibrium, without using dynamic programming. This is a step towards constucting a simple proof that V exists. However, at this point an issue arises: although we know in any equilibrium that $\phi_t = G\left(\phi_{t+1}\right)$, for dynamic programming purposes we would like to know $\phi_{t+1} = \Phi\left(\phi_t\right)$, and G may not be invertible. Our strategy is to restrict attention to equilibria where $\phi_{t+1} = \Phi\left(\phi_t\right)$ and Φ is continuous. Obviously this includes all steady state equilibria, all possible equilibria in the case where G is invertible, and many other dynamic equilibria, but it does not include all possibilities. First note that any equilibrium involves selecting an initial price ϕ_0, or equivalently q_0 since we can invert $\phi_0 = \phi(q_0)$ by (6), and then selecting future values from the correspondence $\phi_{t+1} = G^{-1}(\phi_t)$. We impose only that the selection ϕ_{t+1} from $G^{-1}(\phi_t)$ cannot vary with time or the value of ϕ_t.

That is, while the value ϕ_{t+1} obviously varies with ϕ_t, the rule for choosing which branch of G^{-1} from which to select ϕ_{t+1} is assumed to be constant. We know that this is possible for a large class of dynamic equilibria; e.g., one can always use the rule "select the lowest branch of G^{-1}" and construct equilibria where $\phi_t \to 0$ from any initial ϕ_0 in some interval $(0, \bar{\phi}_0)$. While we may not pick up all possible equilibria given our restriction, we pick up a lot. And we emphasize that the purpose of this restriction is limited: we already know that $\beta\phi_{t+1} \leq \phi_t$ for all t and that F is degenerate in any equilibrium; all we are doing here is trying to guarantee $\phi_{t+1} = \Phi(\phi_t)$ where Φ is continuous in order to prove the existence of the value function V in order to use dynamic programming (and for steady states, there is no problem).

In any case, even given $\phi_{t+1} = \Phi(\phi_t)$ where Φ is continuous, we still need to bound ϕ. We do this with M constant, but the arguments are basically the same when M is varying over time if we work with real balances.

Lemma 6 *Assume* $\sup U(X) > \overline{V} \equiv \frac{u(q^*)+U(X^*)}{1-\beta}$. *Then in any equilibrium* ϕ *is bounded above by* $\overline{\phi} = \overline{z}/M$, *where* $U(\overline{z}) = \overline{V}$.

Proof. Clearly lifetime utility V in any equilibrium is bounded by $\bar{V}$. Consider a candidate equilibrium with $\phi M > \bar{z}$ at some date. In the candidate equilibrium, an individual with $m = M$ would want to deviate by trading all his money for general goods since $U(\phi M) > \overline{V}$. Hence, ϕM is bounded above by $\bar{z}$. ■

Now we can we verify the existence and uniqueness of the value function.

Lemma 7 *Let* $S = \mathbb{R}\times \left[0, \overline{\phi}\right]$ *with* $\overline{\phi}$ *defined as in Lemma 6, and consider the metric space given by* $\mathbb{C} = \{\hat{v} : S \to \mathbb{R} \mid \hat{v}$ *is bounded and continuous*$\}$ *together with the sup norm,* $\|\hat{v}\| = \sup |\hat{v}(m, \phi)|$. *Define*

$$\mathbb{C}' = \left\{ \hat{V} : S \to \mathbb{R} | \hat{V}(m, \phi) = \hat{v}(m, \phi) + \phi m \text{ for some } \hat{v} \in \mathbb{C} \right\}.$$

Let $\Phi : [0,\overline{\phi}] \to [0,\overline{\phi}]$ be a continuous function, and define the operator $T : \mathbb{C}' \to \mathbb{C}'$ by

$$\left(T\hat{V}\right)(m,\phi) = \sup_{m_{+1}} \left\{ v(m,\phi) + \phi m - \phi m_{+1} + \beta \hat{V}[m_{+1}, \Phi(\phi)] \right\}$$

where $v(m,\phi)$ is defined in (9). Then T has a unique fixed point $V \in \mathbb{C}'$.

Proof. First we show $T : \mathbb{C}' \to \mathbb{C}'$. For every $\hat{V} \in \mathbb{C}'$ we can write

$$\left(T\hat{V}\right)(m,\phi) = v(m,\phi) + \phi m + \sup_{m_{+1}} w[m_{+1}, \Phi(\phi)]$$

where $w[m_{+1}, \Phi(\phi)] = \beta \hat{v}[m_{+1}, \Phi(\phi)] + \beta\phi m_{+1} - \phi m_{+1}$ for some $\hat{v} \in \mathbb{C}$. Since $\hat{v}$ is bounded, there exists a $\overline{m}$ such that $\beta w[0, \Phi(\phi)] > \beta w[m_{+1}, \Phi(\phi)]$ for all $m_{+1} \geq \overline{m}$. Therefore,

$$\sup_{m_{+1}} w[m_{+1}, \Phi(\phi)] = \max_{m_{+1} \in [0,\overline{m}]} w[m_{+1}, \Phi(\phi)],$$

and the maximum is attained. Using $w^*(\phi)$ to denote the solution, we have $T\hat{V}(m,\phi) = v(m,\phi) + w^*(\phi) + \phi m \in \mathbb{C}'$, since $w^*(\phi) \in \mathbb{C}$ by the Theorem of the Maximum and $v(x,\phi) \in \mathbb{C}$ from the bargaining solution.

We now show T is a contraction mapping. Define the norm $\left\|\hat{V}_1 - \hat{V}_2\right\| = \sup|\hat{v}_1(m,\phi) - \hat{v}_2(m,\phi)|$ and consider the metric space $(\mathbb{C}', \|\cdot\|)$. Fix $(m,\phi) \in S$. Letting $m^i_{+1} = \arg\max_{m_{+1} \in [0,\overline{m}]} \left\{ \beta \hat{V}_i[m_{+1}, \Phi(\phi)] - \phi m_{+1} \right\}$, we have

$$\begin{aligned} T\hat{V}_1 - T\hat{V}_2 &= \left\{ \beta \hat{V}_1\left[m^1_{+1}, \Phi(\phi)\right] - \phi m^1_{+1} \right\} - \left\{ \beta \hat{V}_2\left[m^2_{+1}, \Phi(\phi)\right] - \phi m^2_{+1} \right\} \\ &\leq \beta \left| \hat{V}_1\left[m^1_{+1}, \Phi(\phi)\right] - \hat{V}_2\left[m^1_{+1}, \Phi(\phi)\right] \right| \leq \beta \left\| \hat{V}_1 - \hat{V}_2 \right\|. \end{aligned}$$

Similarly, $T\hat{V}_2 - T\hat{V}_1 \leq \beta \left\|\hat{V}_1 - \hat{V}_2\right\|$. Hence $\left|T\hat{V}_2 - T\hat{V}_1\right| \leq \beta \left\|\hat{V}_1 - \hat{V}_2\right\|$. Taking the supremum over (m,ϕ) we have $\left\|T\hat{V}_1 - T\hat{V}_2\right\| \leq \beta \left\|\hat{V}_1 - \hat{V}_2\right\|$, and T satisfies the definition of a contraction.

We now argue that $(\mathbb{C}', \rho)$ is complete. Clearly, if $\hat{V}_n(m,\phi) = \hat{v}_n(m,\phi) + \phi m$ is a Cauchy sequence in $\mathbb{C}'$ then $\{\hat{v}_n(m,\phi)\}$ is a Cauchy sequence in $\mathbb{C}$. Since $(\mathbb{C}, \|\cdot\|)$ is complete (see, e.g., Stokey and Lucas [1989], Theorem 3.1), $\hat{v}_n \to v \in \mathbb{C}$. If we set $V = v + \phi m$ it is immediate that $\hat{V}_n \to V \in \mathbb{C}'$. Therefore $(\mathbb{C}', \rho)$ is complete. It now follows from the Contraction Mapping Theorem (see, e.g., Stokey and Lucas [1989], Theorem 3.2) that T has a unique fixed point $V \in \mathbb{C}'$. ■

The final thing to do is fill in some details for the proof of Proposition 1.

Proof of Proposition 1: Most of what is stated follows directly from the analysis in the text, but two details need to be addressed. First, consider the uniqueness of the solution to $e(q) = 1 + \frac{1-\beta}{\alpha\sigma\beta}$ for a general θ. This would follow if $e'(0) < 0$. Given the normalization $c(q) = q$,

$$e(q) = \frac{(\theta u' + 1 - \theta)^2 u'}{(\theta u' + 1 - \theta)u' - \theta(1-\theta)(u-c)u''}.$$

Therefore e' takes the same sign as

$$\begin{aligned} D_1 \;\; = \;\; & \left[(\theta u' + 1 - \theta)u' - \theta(1-\theta)(u-c)u''\right]\left[(\theta u' + 1 - \theta)u'' + 2\theta u'u''\right] \\ & - u'(\theta u' + 1 - \theta)\left[(\theta u' + 1 - \theta)u'' + \theta u'u''\right] \\ & + u'(\theta u' + 1 - \theta)\left[\theta(1-\theta)(u'-1)u'' + \theta(1-\theta)(u-c)u'''\right]. \end{aligned}$$

After simplification we arrive at

$$\begin{aligned} D_1 \;\; = \;\; & -2\theta^2(1-\theta)(u-c)u'u''^2 + \theta(\theta u' + 1 - \theta)\left[u' + (1-\theta)(u'-1)u'u''\right] \\ & - \theta(1-\theta)(\theta u' + 1 - \theta)(u-c)u''^2 + \theta(1-\theta)(\theta u' + 1 - \theta)(u-c)u'u'''. \end{aligned}$$

Since $q^s < q^*$, we have $u' > 1$ and all but the final term are unambiguously negative. If $\theta = 1$ this term vanishes and $D_1 < 0$. For any $\theta \in (0,1)$, if u' is log-concave then $u'u''' < u''$, and this term is bounded by the previous term and so again we have $D_1 < 0$.

Second, we show that e shifts up with an increase in θ at the solution to $e(q) = 1 + \frac{1-\beta}{\alpha\sigma\beta}$. To begin, note that $\partial e / \partial \theta$ takes the same sign as

$$\begin{aligned}
D_2 &= 2(u' - 1)\left[(\theta u' + 1 - \theta)u' - \theta(1-\theta)(u-c)u''\right] \\
&\quad -(\theta u' + 1 - \theta)\left[u'(u'-1) - (1-2\theta)(u-c)u''\right] \\
&= (\theta u' + 1 - \theta)(u'-1)u' - (\theta u' - 1 + \theta)(u-c)u''
\end{aligned}$$

Now rearrange $e(q) = 1 + \frac{1-\beta}{\alpha\sigma\beta}$ as

$$(u-c)u'' = \frac{(\theta u' + 1 - \theta)u'}{\theta(1-\theta)\left(1 + \frac{1-\beta}{\alpha\sigma\beta}\right)}\left(\frac{1-\beta}{\alpha\sigma\beta} + \theta - \theta u'\right)$$

Substituting this into D_2 and simplifying, at $e(q) = 1 + \frac{1-\beta}{\alpha\sigma\beta}$ we see that D_2 takes the same sign as $\frac{1-\beta}{\alpha\sigma\beta}(1-\theta)^2 + \theta^2 u'(u'-e)$. The desired result follows if we can show $u' \geq e$, which is easy to establish. ■

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