Gresham's Law versus Currency Competition[*]

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Abstract

We develop a dual currency search model where agents can hold currency portfolios to buy goods, and analyze an agent's choice to spend safe dollars or risky home currency for internal trade. We focus on two equilibria: a *currency competition* equilibrium, in which the 'good' currency (dollars) is spent first and the 'bad' (risky home) currency is kept for later purchases, and a *Gresham's Law* equilibrium in which agents do the reverse. We prove that for the Gresham's Law equilibrium to prevail, trading frictions and the home currency risk must be small. Otherwise, extensive currency substitution occurs and the currency competition equilibrium prevails. Interestingly, because transaction velocity is endogenous, we demonstrate that as the home currency risk rises, currency substitution causes a decline in the transaction velocity of the bad currency while increasing it for the good currency.

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"The full-bodied coins that are the pride of Athens are never used while the mean brass coins pass hand to hand." - Aristophanes, "The Frogs"

"[In the post-World War I European hyperinflation,] the lack of a stable domestic means of payments was a serious inconvenience...and foreign currencies therefore came to be desired...as a means of payment...Thus, in advanced inflation, "Gresham's Law" was reversed: good money tended to drive out bad..." - League of Nations (1946, p. 48)

1   Introduction

Monetary history is replete with examples in which different objects are accepted as media of exchange – gold and silver coins, several privately issued bank notes, or two fiat currencies. Of particular interest to economists is what happens to transaction patterns when one of the monies is viewed as being 'superior' in some way to the others. As illustrated by Aristophane's quote, a common belief has been that the inferior currency would circulate more widely than the superior currency. We know this as Gresham's Law – bad money drives out good money. In this situation, buyers holding both monies prefer to spend the bad money and hold onto the good one for future consumption. The implication of this is that bad currency circulates more widely with the limiting case being that only the bad currency circulates.[1]

However, the second quote is based on an observation that is contrary to Gresham's Law. Even today, a common occurrence in developing countries is for a 'good' foreign currency (dollars) to circulate more widely than the 'bad' domestic currency.[2] Here, agents spend the good currency and hold onto the bad currency for future consumption. Consequently, the good currency circulates more widely with the *limiting* case being that only the good currency circulates. Hayek (1976) argued that this spending pattern was the logical outcome of 'currency competition', in which case good money drives out bad money.

While Gresham's Law and currency competition are generally accepted by economists as describing rational spending behavior, these two outcomes imply contradictory transaction patterns.

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[1] With commodity monies, agents could also 'ship out' the good currency or 'melt it down'. Episodes involving such actions have been described in Rolnick and Weber (1986) and Sargent and Smith (1997). Obviously, fiat currencies cannot be 'melted down' but they can be imported or exported for use as media of exchange (Peterson, 2001).

[2] This phenomenon is commonly known as currency substitution.

Thus, a fundamental challenge for monetary economists is to determine, within a common framework, conditions under which one currency circulates more widely than another. In short, the question we ask is as follows: if two currencies are generally accepted in trade, when will agents spend the bad currency and hold the good currency for future consumption? When will they do the opposite?

To address these questions, we need to construct a trading environment in which money is essential as a medium of exchange and agents' transaction patterns are carefully specified. Consequently, we conduct our analysis using a search-theoretic model of money, in the tradition of Kiyotaki and Wright (1993). We build a one country, two-currency model in which agents are allowed to hold multiple units of fiat monies, and prices are endogenously formed. The two currencies are fundamentally different in that one is assumed to be 'risky', whereas the other is not. We call the safe currency 'good', and the risky currency 'bad'. In equilibrium, spending patterns and the distribution of prices are driven by the relative riskiness of the currencies and not by ad-hoc transactions costs or institutional restrictions.

We start by analyzing equilibria in which agents prefer to spend the good currency first and the bad currency later. We refer to this as the 'currency competition' equilibrium since the good currency is used more frequently as a medium of exchange. Our analysis shows that this equilibrium will tend to arise when the risk on the domestic currency is sufficiently large and the trading environment does not function well, i.e. trading frictions are severe.

We then study a 'Gresham's Law' type of equilibrium in which agents choose the opposite spending pattern. They spend the bad currency first, and hold on to the good currency for future purchases. While this may appear to be the most obvious strategy for the buyer, our analysis reveals that, in fact, this equilibrium is harder to support. The reason is that while getting rid of the bad (risky) currency first makes sense for the buyer, it effectively transfers the risk onto the seller who will not accept the risk without being compensated. Compensation takes the form of higher prices in terms of the risky currency, which in turn lowers current consumption for the buyer. If the currency risk is large enough, the buyer gets so little for the bad currency that he would prefer to spend the good currency first. We show that this Gresham's Law equilibrium will only tend to exist if the risk on the bad currency is sufficiently low and the trading environment is well-functioning, i.e., trading frictions are very low.

Despite its level of abstractness, we believe that the model captures key aspects of many developing economies where the dollar circulates alongside the domestic currency. Our analysis suggests that the level of 'dollarization' can be kept low as long as the domestic currency risk is low and the domestic trading environment is well functioning. However, should currency risk get out of hand or the economic trading environment break down, a high degree of dollarization will be the outcome. Our model generates an equilibrium distribution of real exchange rates. We can show that an increase in domestic currency risk leads to a depreciation in the real value of the domestic currency and an increase in the dispersion of observed real exchange rates. Finally, we show numerically that as the risk on the domestic currency increases, its transaction velocity falls while that of the foreign currency increases.

The structure of the paper is as follows. Section 2 contains a discussion of related literature. Section 3 describes the economic environment. Section 4 contains our definition of an equilibrium. Section 5 examines the currency competition equilibrium. Section 6 examines the Gresham's Law equilibria. Section 7 contains numerical analysis. Section 8 contains concluding comments.

2 Related Literature

A substantial amount of research has looked at environments with competing currencies. Giovannini and Turtleboom (1994) provide a good survey of this line of research and the types of models used. They group most research on currency substitution into three classes of models: 1) cash-in-advance models, 2) transaction cost models and 3) ad-hoc models. The main problem with all of these models is that they do not have a fundamental role for money as a medium of exchange. Hence, arbitrary restrictions regarding the use of money and/or ad-hoc transaction costs from using a particular currency must be employed. A more preferred approach would be to construct a model in which money has an explicit role, in an environment in which trading frictions are not a function of the currency used. In short, we want a level trading field.

Search theoretic models of money have these properties and have been used to study the use of multiple currencies as media of exchange.[3] Search models have been used to study currency

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[3]For example, Aiygari, Wallace and Wright (1996) examine rate of return dominance and coexistence of dual media of exchange. Matsuyama, Kiyotaki and Matsui (1993), Wright and Trejos (2001), Kocherlakota and Krueger (1999), Camera and Winkler (2000) look at international versus national currencies. Zhou (1997), Waller and Curtis (2001), Craig and Waller (2001), Head and Shi (2000), Ravikumar and Wallace (2001) generate currency exchange.

substitution and Gresham's Law [Velde, Weber and Wright (1999), Renero (1999) and Burdett et al. (2001)] but this work has focused almost exclusively on the use of commodity money in order to distinguish between 'superior' and 'inferior' currencies, and not *fiat* currencies. Furthermore agents are generally allowed to carry only one unit of money when conducting trade, i.e. they cannot hold currency portfolios, which makes it impossible to study equilibria in which, in a match, an agent must choose to spend one currency or the other when buying goods.[4] Furthermore, it is not possible to study how changes in the relative inferiority of the domestic currency alters transaction patterns and circulation of each currency. We believe that transaction patterns are crucial to understanding the process of currency competition. Thus, the model in this paper is the first to study currency competition and Gresham's Law in this fashion.

3   Economic Environment

The environment is based on the standard monetary search model. There is a continuum of infinitely lived agents uniformly located on the unit interval who specialize in consumption and production of goods and services. There is a continuum of good types defined on the unit circle. Agents specialize in production and consumption. Specifically, an agent can produce only one type of good, but consume a subset of good types. When producing the quantity $q > 0$, the agent incurs a linear production cost measured in units of utility given by $c(q) = q$. When consuming q units of a desired consumption good the agent obtains utility $u(q)$, with $u'(q) > 0$, $u''(q) < 0$ and $u'(0) = \infty$.

Agents meet bilaterally and at random via a Poisson process with arrival rate $\alpha > 0$. The matching process is such that, contingent on meeting, there is probability x of single coincidence of wants, and xy of double coincidence. We set $y = 0$, to rule out barter so that agents must resort to alternative means of conducting trade, such as money. To make money essential, we assume away the existence of alternative payments systems or financial intermediaries. Agents can trade with either currency and, most importantly, the trading frictions are the same for both currencies. We assume the transaction costs are identical for each currency in terms of its use in trade.

Agents are initially randomly endowed with indivisible units of two types of fiat money, which we will refer to as the foreign (or, the dollar) and home currency as a way of distinguishing the two.

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[4]While Craig and Waller (2001) allow agents to hold currency portfolios, the analysis is all done numerically and the transaction patterns are stunningly complex. Thus, to make any analytical progress, the model used in Craig and Waller must be reduced dramatically.

Each is in constant per capita supply, M_f and M_h, where the subscript f refers to foreign and h denotes home. An individual can hold at most N units of money in total. In order to make one of the currencies 'inferior', we allow the currencies to be fundamentally different with respect to their purchasing power risk. Specifically, we proceed as in Li (1995) by assuming that agents can have their holdings of home currency randomly confiscated by the government.

An agent meets with the government with arrival rate α. Upon meeting an agent holding the home currency, the government randomly confiscates all of the agent's home currency holdings with probability $\tau \in [0, 1]$. Confiscated currency holdings are destroyed immediately. The government consumes all goods and services but does not produce them. For this reason, conditional on meeting a seller, the government buys goods from the agent with probability $\eta \in [0, 1]$, paying with a new unit of home currency.[5] While highly stylized, the randomness of confiscation captures the idea that the home currency is risky and those holding it are prone to sudden losses of purchasing power. Because of this difference in the risk, we refer to the foreign currency as the 'good' currency and the home currency as the 'bad' currency.

4   Symmetric Stationary Equilibria

We study stationary rational expectations equilibria, where symmetric Nash strategies are adopted, and identical agents use identical time-invariant pure strategies. Furthermore we study equilibria where the beliefs over strategies and traded quantities are identical across individuals, and each agent correctly evaluates the potential gains from trade in all matches.

Agents must use money to conduct trade. We examine the case in which both currencies are fully acceptable media of exchange.[6] Agents thus can hold a 'portfolio' of currencies. To simplify the analysis of the transaction patterns, we let $N = 2$. The reason for this assumption is two-fold. First, there is only one 'diversified' currency portfolio consisting of one unit of each currency. This allows us to focus our entire analysis of spending behavior on the actions of these portfolio holders. Second, no pure currency trades will arise, i.e., currency does not trade for currency.[7] This allows

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[5]The government has three parameters under its control, τ, η, M_h. Two of these are free parameters while the third must adjust to maintain a balanced budget constraint. We set τ and M_h and let η be endogenously determined.

[6]There is always an equilibrium in which one or both currencies are not accepted.

[7]If the two currencies have different values, then one-for-one currency trades will not exist. With an upper bound of 2, the only remaining trade is a 2 for 1 trade. However, these trades require that the two traders 'swap' their entire portfolios, which has to make one of them worse off. So these trades do not occur either. Thus, with an upper bound

us to focus on goods trades only and ignore nominal exchange rate determination in pure currency trades. However, there is the possibility for currency to trade for the other currency plus some goods, as in Aiyagari, Wallace and Wright (1996). We rule out these trades for two important reasons. First, allowing them can generate equilibria in which two identical currencies trade at different values simply due to beliefs.[8] In order to focus on differences in currency values arising strictly from 'fundamentals', such as currency risk, this potential source of extrinsic valuation needs to be controlled for. Preventing these trades is one way to do it. Second, we cannot obtain any analytical results if these trades are allowed and must resort to numerical methods to study equilibria where these trades occur as is done in Craig and Waller (2001).

Let m_i denote the fraction of agents in the economy holding a currency portfolio i, where $i \in \{0, f, h, 2f, 2h, fh\}$ denotes the composition of the portfolio. For example fh means that the agent has one unit of each currency, $2f$ that she has two units of the foreign currency and so on. As a result m_i must satisfy the following constraints:

$$
\begin{aligned}
1 &= m_0 + m_f + m_h + m_{2f} + m_{2h} + m_{fh} \\
M_f &= m_f + 2m_{2f} + m_{fh} \\
M_h &= m_h + 2m_{2h} + m_{fh}
\end{aligned}
\tag{1}
$$

where $M_f + M_h < 2$ since $N = 2$. In a stationary equilibrium $\dot{m}_i = 0$ for all i.[9] Furthermore, to keep the per capita stock of home currency constant the outflows must be offset by the inflows:

$$
\tau(m_h + 2m_{2h} + m_{fh}) = \eta \left[m_0 + m_f + m_h \right].
\tag{2}
$$

The terms of trade are endogenously formed. Agents with money can be buyers or sellers in a bilateral match, depending on their trading partner. Agents with no currency, however, can only be sellers since all exchange must be quid-pro-quo, barter is not feasible, and there is no credit. Note, however, that since $N = 2$, only those agents with portfolios $0, h$, and f can be sellers Agents with two-unit portfolios can only be buyers; we denote their proportion in the economy by $\mu = m_{2f} + m_{2h} + m_{fh}$. The trading mechanism is assumed to be based on take-it-or-leave-it

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of 2 units, pure currency trades will not occur.

[8] For example, suppose two currencies are identical except for their colors. If agents believe blue currency is more valuable than red currency, then an equilibrium consistent with this belief can be supported for some parameter values (see Aiyagari, Wallace and Wright (1996) or Cavalcanti (2000)).

[9] The laws of motion depend on the transaction pattern and are described in a later section.

bargaining protocol. Specifically, when a buyer meets a seller, he offers the seller a trade of d units of currency for the quantity q of goods. The seller can accept or reject. Thus, the optimal offer pair (d, q) is such that the seller is left indifferent between accepting and rejecting it.[10] Consequently, the seller gets zero net surplus in all trades, and always accepts the currency. When the government buys goods it also makes a take-it-or-leave-it offers.

To define prices one must specify the equilibrium transaction pattern. We focus on the one studied by Camera and Corbae (1999), in which agents only spend one unit of money per transaction, i.e. $d = 1$. In this case the price in a transaction is given by $1/q$. While there are many transaction patterns that one can consider in this setting, we choose this particular pattern because it is the simplest way to analyze the agent's choice of spending one currency or the other.

We want to determine the conditions under which an agent holding a diversified portfolio will prefer to spend his unit of the good (foreign) currency rather than the bad (home) currency, and vice versa. The choice of which currency to trade is complicated because it is contingent on the seller's money holdings. For example, sellers holding a dollar are willing to produce a different amount for a second dollar than will a seller holding a unit of home currency. Restricting attention to a representative buyer with portfolio fh, let $p_i \in [0, 1]$ denote the probability that he chooses to spend currency f when he is matched to a seller with portfolio $i \in \{0, h, f\}$. With the complementary probability, $1 - p_i$, he spends his unit of home currency. To describe this buyer's spending strategy, when $d = 1$, we use the vector $p = (p_0, p_f, p_h)$. We denote by p^* the equilibrium strategy vector. Note that although we will limit our analysis to pure strategies, there are eight possible equilibrium strategy vectors p^*.

Let V_i denote the value associated with holding portfolio i for a given vector p^*. Furthermore, let q_i^j denote the equilibrium quantity produced by a seller with portfolio i in exchange for one unit of currency $j = f, h$. Under the conjecture that $d = 1$ and buyer-take-all bargaining, V_i must

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[10]Because of the indivisibility of money, d must be an integer. Therefore, the optimal offer pair (d, q) may not maximize the surplus in the match. However, Berentsen, Molico and Wright (2001) show that agents may choose to engage in lotteries over d to improve the expected surplus from trade. Allowing for lotteries would substantially complicate the analysis.

satisfy $V_0 = 0$ and

$$
\begin{aligned}
\rho V_i &= x \sum_{j \in \{0,f,h\}} m_j u(q_j^i) - x(1-\mu)(V_i - V_0) - \tau(V_i - V_0)1_{\{i=h\}} \\
\rho V_{2i} &= x \sum_{j \in \{0,f,h\}} m_j u(q_j^i) - x(1-\mu)(V_{2i} - V_i) - \tau(V_{2i} - V_0)1_{\{i=h\}} \\
\rho V_{fh} &= \max_{p_j \in \{0,1\}} x \sum_{j \in \{0,f,h\}} m_j \left[p_j u(q_j^f) + (1 - p_j) u(q_j^h) \right] + x \sum_{j \in \{0,f,h\}} m_j p_j (V_h - V_f) \\
&\quad - x(1-\mu)(V_{fh} - V_f) - \tau(V_{fh} - V_f)
\end{aligned}
\tag{3}
$$

where the indicator function $1_{\{i=h\}} = 1$ (and zero otherwise), and $\rho = r/\alpha$ is the discount factor adjusted by the arrival rate. It measures the severity of the trading frictions in the economy: as ρ goes to zero, frictions vanish. The first term on the right-hand side of each of the value functions in (3), is the expected utility from current consumption matches, i.e. those with sellers who can produce one's desired good. With probability xm_j the agent meets a seller with portfolio j who can produce his desired consumption good which pays off utility $u(q_j^i)$, when currency i is used in the transaction. The second term is the expected value from changing the portfolio from spending (or acquiring) a unit of currency, which occurs with probability $x(1-\mu)$. For holders of the home currency, $i = h$, the third term is the expected loss from having the government confiscate one's holdings of home currency. This occurs with probability τ. For agents holding portfolio fh, the first term represents the expected utility from meeting sellers and acquire consumption goods by spending a unit of currency. The first component in the brackets is the utility from choosing to spend the dollar, with probability p_j, in a match with a seller with portfolio j. The second component is the utility derived from choosing to spend the home currency in that match. The second and third terms are the expected payoffs from changing portfolio states and the last term is the expected loss due to confiscation.

It is useful to manipulate the value functions in (3) in order to show that, for any equilibrium p^*, the value of multiple-unit portfolios can be expressed as linear combinations of the values associated with single-unit holdings. Specifically for homogenous portfolios of currency $i = h, f$

$$
\begin{aligned}
V_i &= \tfrac{A_i}{1-\mu} \left[m_0 u(q_0^i) + m_f u(q_f^i) + m_h u(q_h^i) \right] \\
V_{2i} &= (1 + A_i)V_i
\end{aligned}
\tag{4}
$$

while for a diversified portfolio

$$
V_{fh} = \frac{A_h}{1-\mu} \sum_{j \in \{0,f,h\}} m_j p_j^* \left[u(q_j^f) - u(q_j^h) + V_h - V_f \right] + V_h + A_{fh} V_f
\tag{5}
$$

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where

$$A_h = \frac{x(1-\mu)}{\rho + \tau + x(1-\mu)} < A_f = \frac{x(1-\mu)}{\rho + x(1-\mu)} < A_{fh} = \frac{\tau + x(1-\mu)}{\rho + \tau + x(1-\mu)} < 1$$

such that as $\rho \to 0$, then A_f, $A_{fh} \to 1$ while $A_h < 1$ for $\tau > 0$. It is immediate that, in a monetary equilibrium, the expected lifetime utility of any portfolio is bounded below by zero. It is also concave in the size of currency holdings, whereby $V_{2i} \leq 2V_i$ and $V_{fh} \leq V_h + V_f$ for all parameters and any p^*. Furthermore, $(V_{2f} - V_f)/V_f = A_f > (V_{2h} - V_h)/V_h = A_h$, i.e. the percentage gain in expected lifetime utility from acquiring a second dollar is greater than the percentage gain in utility from acquiring a second unit of the risky home currency.

In equilibrium, the quantities exchanged in the matches are such that the cost of producing equals the expected utility from acquiring a unit of currency, that is

$$
\begin{array}{ccc}
\text{Foreign} & & \text{Home} \\
q_0^f = V_f & & q_0^h = V_h \\
q_f^f = V_{2f} - V_f & & q_f^h = V_{fh} - V_f \\
q_h^f = V_{fh} - V_h & & q_h^h = V_{2h} - V_h
\end{array}
\tag{6}
$$

Although the buyer with portfolio fh can choose among eight possible pure strategies, we concentrate on the two opposing cases in which the buyer always spends the dollar, $p^* = (1,1,1)$, or he always spend the home currency, $p^* = (0,0,0)$. We study the other pure strategy equilibria numerically.

We think of the $p^* = (1,1,1)$ case as corresponding to 'currency competition' – agents prefer to spend the good currency rather than the bad currency when a trading opportunity arises. What is interesting about this strategy is that the buyer gives up the safe currency and chooses to hold onto the risky currency rather than dumping the risky currency when the opportunity arises. We consider the $p^* = (0,0,0)$ equilibrium to be a 'Gresham's Law' equilibrium because the buyer spends the bad currency when given the opportunity and hoards the good currency. These opposing strategies are appealing in that they are non-discriminatory, i.e. all sellers are offered the same currency.

To prove existence of an equilibrium, we follow the approach of Camera and Corbae (1999). Given the conjecture that $d = 1$ and a strategy vector p^* are optimal, we derive necessary conditions such that the conjectured strategies are individually optimal. Then we solve for the equilibrium value functions, quantities, and distributions of portfolios, providing conditions sufficient to satisfy individual optimality, in term of the parameters of the model.

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4.1   Individual optimality conditions

To determine the conditions under which the conjectured transaction pattern is individually optimal we must do the following. First, for any given p^*, $d = 1$ is optimal if agents choose to spend at least one unit of currency but no more than one unit. This implies that for those buyers holding two units of currency matched to a seller with no currency, the trade surplus from spending one unit is greater than that from spending both units. Since sellers holding one unit of currency cannot accept two, due to the inventory constraint, the only meetings that matter are those between two-unit buyers and sellers holding no currency. With three types of two-unit buyers there are three optimality conditions that must be satisfied given by:

$$
\begin{aligned}
u(V_f) + V_f - V_{2f} &> u(V_{2f}) + V_0 - V_{2f} &&(2f \text{ buyer}) \\
u(V_h) + V_h - V_{2h} &> u(V_{2h}) + V_0 - V_{2h} &&(2h \text{ buyer}) \\
\max\left\{u(V_f) + V_h, u(V_h) + V_f\right\} - V_{fh} &> u(V_{fh}) + V_0 - V_{fh} &&(fh \text{ buyer})
\end{aligned}
\tag{7}
$$

Second, it must be the case that the trade surplus buyers receive from spending one unit of currency is larger than the payoff from walking away. It is straightforward to show that if buyers holding one unit of currency choose not to walk away, 2-unit buyers will not walk away either. Since 'rich' sellers (those holding currency) produce less than 'poor' sellers (those with no currency), if buyers with one unit of currency buy from rich sellers, they will also buy from poor sellers. Since there are two poor buyer states, f and h, and two rich seller states, f and h, then the condition to spend at least one unit generates four optimality constraints and are given by:

$$
\begin{aligned}
u(V_{fh} - V_f) + V_0 - V_h &> 0 &&(h \text{ buyer}, f \text{ seller}) \\
u(V_{2h} - V_h) + V_0 - V_h &> 0 &&(h \text{ buyer}, h \text{ seller}) \\
u(V_{2f} - V_f) + V_0 - V_f &> 0 &&(f \text{ buyer}, f \text{ seller}) \\
u(V_{fh} - V_h) + V_0 - V_f &> 0 &&(f \text{ buyer}, h \text{ seller})
\end{aligned}
\tag{8}
$$

Finally, under the conjecture that $d = 1$, we must verify that a buyer at portfolio fh chooses to spend either the dollar or the home currency when he meets a seller. Consequently, the trading surplus from spending one currency must be larger than the trading surplus from spending the other currency. Since there are three sellers, $\{0, f, h\}$, there are three conditions that need to be

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satisfied in order for p to be optimal:

$$p_0 = 1 \quad \Leftrightarrow \quad u(V_f) + V_h - V_{fh} > u(V_h) + V_f - V_{fh}$$
$$p_f = 1 \quad \Leftrightarrow \quad u(V_{2f} - V_f) + V_h - V_{fh} > u(V_{fh} - V_f) + V_f - V_{fh} \tag{9}$$
$$p_h = 1 \quad \Leftrightarrow \quad u(V_{fh} - V_h) + V_h - V_{fh} > u(V_{2h} - V_h) + V_f - V_{fh}$$

and $p_i = 0 \; \forall i$, if the corresponding inequality is reversed.

Despite the large number of inequalities to be satisfied, it turns out that there is one of particular interest, namely the first one in (9). It describes the fh buyer's decision to offer a dollar to a seller with no money. Rewrite it as

$$S(V_f) \equiv u(V_f) - V_f > u(V_h) - V_h \equiv S(V_h)$$

This expression has a simple and intuitive interpretation. $u(V_f)$ is the utility gain from spending a dollar and consuming $q_0^f = V_f$. V_f is also the value of the foregone portfolio state f. Thus, the net gain from spending the good currency is $S(V_f) \equiv u(V_f) - V_f$. Similarly, $S(V_h)$ is the net gain from using the bad currency. Thus the agent has to compare the two strategies and chooses to spend the dollar if the net gain is larger. Note that the two sides of the inequality evaluate the same function at different points. Thus, the functional form of preferences matters in determining whether spending the dollar is optimal, and the relative value of the two currencies is the critical element. Specifically, there are two cases to consider depending on whether the net gain is monotonically increasing in V or if it decreases as V becomes large.

If $S(V)$ is monotonically increasing, then an immediate result for $p^* = (1, 1, 1)$ to individually optimal is that $V_f > V_h$. In short the dollar must be more valued than the home currency. This makes intuitive sense because the home currency is risky. By making a purchase with the home currency the buyer transfers the risk to the seller. However, the seller will not accept the risk unless he is compensated for it. A natural way to compensate the seller is to ask for a smaller quantity of goods. Since agents prefer current to future consumption, and since the net gain increases monotonically in the value of the transaction, the buyer will prefer to make a dollar purchase, whenever possible. On the other hand, $p^* = (0, 0, 0)$ can be an equilibrium only if $V_h > V_f$. This implies that the home currency, despite its fundamental risk, has greater purchasing power than the dollar. In short, not only does the seller accept a risky currency but he chooses to compensate the buyer in the process by producing more today! It is hard to believe that this behavior can be supported as an equilibrium on a large region of the parameter space.

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If $S(V)$ is not monotonically increasing, then it is possible that $p^* = (0, 0, 0)$ can be supported when $V_h < V_f$. Intuitively this is because even if the dollar buys more goods, there is a high opportunity cost in spending it. Consequently, $S(V_f)$ could be very small if the net gain exhibits decreasing returns for high value transactions. In this case, despite the fact that it buys less today, it is better to spend the bad currency and hold on to the good currency for future purchases. Doing so raises the net gain since, the lower opportunity cost more than compensates for the drop in consumption.

To illustrate how the form of preferences affects this surplus and the possible equilibrium transaction patterns, we will consider two forms of utility. The first is given by $u(q) = q^\sigma + q$, $0 < \sigma < 1$. This function exhibits decreasing relative risk aversion and implies that in a match between an fh buyer and a seller with no money, the net gain, $S(V) = V^\sigma$, is monotonically increasing in V. Second, we consider the CRRA function $u(q) = q^\sigma$, a specification common in many search-theoretic models of money. This implies that $S(V) = V^\sigma - V$ is a hump-shaped function that is zero at $V = 0, 1$ and has a unique maximum at $\hat{V} = \sigma^{\frac{1}{1-\sigma}} < 1$.

Definition. *A steady-state dual currency monetary equilibrium is a set of value functions satisfying $V_h, V_f > 0$ and (3), a distribution of portfolios satisfying (1)-(2) and $\dot{m}_i = 0$, prices given by (6), and a set of strategies d and p satisfying (7)-(9).*

5 The Currency Competition Equilibrium: $p^* = (1, 1, 1)$

In this section we determine conditions under which the buyer with portfolio fh decides to spend the dollar in all transactions. Under the conjecture that $p^* = (1, 1, 1)$, (5) becomes

$$V_{fh} = V_f + A_h V_h \tag{10}$$

so that, using (6)

$$q_h^f = V_f - (1 - A_h)V_h \quad \text{and} \quad q_f^h = A_h V_h \tag{11}$$

where $q_h^f > 0$ only if $(1 - A_h)V_h < V_f$.

Given the conjectured pattern of transactions, the distribution of portfolio holdings must be feasible, i.e. it must satisfy (1)-(2) and $m_i \in (0, 1)$. In the steady state it must also be stationary,

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i.e. $\dot{m}_i = 0$ for all portfolios i. Due to the linear dependency of m_i we need to consider only three flow conditions. We focus on

$$\dot{m}_{2f} = m_f (m_f + m_{fh}) - m_{2f} (m_0 + m_h) = 0 \qquad (12)$$

$$\dot{m}_{2h} = x[m_h^2 - m_{2h} (m_0 + m_f)] + \eta m_h - \tau m_{2h} = 0 \qquad (13)$$

$$\dot{m}_{fh} = x[m_f m_{2h} + m_h m_{2f} + 2m_h m_f - m_{fh} (m_0 + m_f)] + \eta m_f - \tau m_{fh} = 0 \qquad (14)$$

In the appendix we derive sufficient conditions for existence of a stationary distribution and show that it is unique.

Using (4), (10), and (11), we obtain:

$$V_h = \frac{A_h \left[m_0 u(V_h) + m_f u(A_h V_h) + m_h u(A_h V_h) \right]}{1 - \mu} \qquad (15)$$

$$V_f = \frac{A_f \left[m_0 u(V_f) + m_f u(A_f V_f) + m_h u(V_f - (1 - A_h) V_h) \right]}{1 - \mu} \qquad (16)$$

where (16) is defined only for $V_f > (1 - A_h) V_h$. Note that $V_h = V_f = 0$ solves (15) and (16); this is the non-monetary equilibrium.

Let (V_f^*, V_h^*) denote a positive fixed point of the map given by (15) and (16). We discuss existence of positive fixed points in the next lemma (all proofs of lemmas and propositions are in the appendix).

Lemma 1. *Suppose $d = 1$ and $p^* = (1, 1, 1)$ is a dual currency monetary equilibrium. Then there always exists a unique V_h^*, and $V_f^* = V_h^*$ whenever currency risk is absent. If currency risk is present, then there can be at most two distinct and mutually exclusive cases:*

(i) $V_f^* > V_h^*$, which always exists and, if ρ is sufficiently small, satisfies $\frac{V_f^*}{V_h^*} \leq \frac{1 - A_h}{1 - A_f}$

(ii) $V_f^* < V_h^*$ which may not exist and, in particular, does not exist either if τ is sufficiently large or if ρ is sufficiently small.

The lemma shows that if there is no fundamental difference between the two currencies ($\tau = 0$) and $p^* = (1,1,1)$ and $d = 1$ is individually optimal, then the only monetary equilibrium is such that both currencies are identically valued. Hence, the fh buyer would be indifferent between spending the home money relative to the dollar and could choose any $p_i \in [0,1]$. When $\tau > 0$, however, this is never the case. This is a general result which holds for any set of concave preferences.

Note that $V_h = 0$ is also a solution to (15). Given $V_h = 0$, it is straightforward to show that there is a unique value $V_f > 0$ that solves (16). In this case, the home currency has no value and would not circulate. This is the limiting case of currency competition – only the good currency circulates. Consequently, it corresponds to the mono-currency equilibrium studied by Camera and Corbae with $N = 2$. Since we are interested in dual currency monetary equilibria, we do not analyze it in detail. While this equilibrium is not the focus of our attention, it is important to recognize that it is one solution under the conjectured transaction pattern. It is also important to recognize that $V_h > V_f = 0$ is never an equilibrium, if $p^* = (1,1,1)$.

One immediate implication of Lemma 1 is that, although currency exchange does not occur, the model generates an equilibrium distribution of real exchange rates (or relative prices). There is more than one relative price in the model since different sellers produce different quantities for different currencies. Let $R_i = q_i^f/q_i^h$ denote the relative price offered by a seller with portfolio $i \in \{0, f, h\}$. This measure gives us the real value of the dollar to a unit of the home currency. Using (11) we obtain

$$R_0 = \frac{V_f}{V_h}$$
$$R_f = \frac{A_f}{A_h} \frac{V_f}{V_h}$$
$$R_h = 1 + \frac{(V_f/V_h) - 1}{A_h}$$

When $\tau = 0$ the distribution of real exchange rates is degenerate, $R_i = 1 \; \forall i$, since $V_f = V_h$ and $\frac{A_f}{A_h} = 1$. When there is some currency risk, however, then $R_0 < R_h < R_f$ for $\frac{V_f}{V_h} < \frac{1-A_h}{1-A_f}$ but $R_0 < R_f < R_h$ if $\frac{V_f}{V_h} > \frac{1-A_h}{1-A_f}$. Thus, as the risk on the home currency increases from zero, the observed spread of real exchange rates increases. Although this is a cross-section of real exchange rates, it loosely corresponds to the idea that greater currency risk leads to an increase in the volatility of observed real exchange rates between the home currency and the dollar.

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5.1 Existence of the $p^* = (1, 1, 1)$ equilibrium

We study existence of the currency competition equilibrium by considering specific preferences. Suppose $u(q) = q^\sigma + q$. We are able to obtain a closed-form equilibrium solution for V_h but not V_f :

$$V_h = \left\{ \frac{A_h \left[m_0 + m_f A_h^\sigma + m_h A_h^\sigma \right]}{1 - \mu - A_h \left[m_0 + (m_f + m_h) A_h \right]} \right\}^{\frac{1}{1-\sigma}} \tag{17}$$

$$V_f = \left\{ \frac{A_f \left[m_0 + m_f A_f^\sigma + m_h (1 - (1 - A_h) (V_h/V_f))^\sigma \right]}{1 - \mu - A_f \left[m_0 + m_f A_f + m_h (1 - (1 - A_h) (V_h/V_f)) \right]} \right\}^{\frac{1}{1-\sigma}} \tag{18}$$

The conditions in (8)-(9) reduce to:

$$\left[\frac{(1 + A_f)^\sigma - 1}{1 - A_f} \right]^{\frac{1}{1-\sigma}} < V_f < \left(\frac{A_f^\sigma}{1 - A_f} \right)^{\frac{1}{1-\sigma}} \tag{19}$$

$$\left[\frac{(1 + A_h)^\sigma - 1}{1 - A_h} \right]^{\frac{1}{1-\sigma}} < V_h < \left(\frac{A_h^\sigma}{1 - A_h} \right)^{\frac{1}{1-\sigma}} \tag{20}$$

$$V_f > V_h \tag{21}$$

$$(1 - A_h) V_h + A_f^\sigma V_f^\sigma > A_h^\sigma V_h^\sigma + (1 - A_f) V_f \tag{22}$$

$$(1 - A_h) V_h + V_f^\sigma > (V_f + A_h V_h)^\sigma \tag{23}$$

The two inequalities (19)-(20) are essentially the same as in Camera and Corbae (1999). In short, the value of holding a unit of currency must be high enough to prevent 2-unit buyers from spending all of their cash but not high enough to prevent expenditures by poor buyers on rich sellers. The new restrictions arising from multiple currencies are (21), (22) and (23). Inequality (21) is the condition for a buyer with a mixed portfolio to spend the dollar rather than the home currency on a poor seller while (22) is the condition that he spends the dollar on a rich seller holding a unit of the home currency. Inequality (23) ensures that a buyer fh only spends the dollar and not both. Equation (21) shows that in equilibrium a necessary condition is that the dollar is more valued than the home currency, $V_f > V_h$. This is a consequence of the net gain being monotonically increasing in V.

With these preferences we provide sufficient conditions for the existence of a unique currency competition equilibrium.

Proposition 1. *Consider $u(q) = q + q^\sigma$ and a stationary distribution supporting the transaction pattern $d = 1$ and $p^* = (1, 1, 1)$. There exist positive values σ_H and ρ_H such that if $\sigma \in (0, \sigma_H)$ and $\rho \in (0, \rho_H)$ then the currency competition equilibrium exists and is unique.*

The intuition for these parameter values is as follows. For sufficiently low trading frictions (small ρ), price dispersion is low, and so agents are always willing to buy now rather than wait for a better deal.[11] Low values of σ imply that the marginal utility of consumption is very high but diminishes rapidly. This ensures that agents spend at least one unit of currency but not two. If trading frictions are low, the buyer holding one unit of each currency is willing to spend the safe foreign currency in all matches and hold onto the risky home currency. This is so because the surplus in trade is increasing in the value of the currency and agents discount future consumption less thereby increasing the value of the dollar relative to the home currency.

If preferences are CRRA, $u(q) = q^\sigma$, the equilibrium (V_f^*, V_h^*) must satisfy

$$V_h = \left\{ \frac{A_h}{1 - \mu} \left[m_0 + m_f A_h^\sigma + m_h A_h^\sigma \right] \right\}^{\frac{1}{1-\sigma}}$$

$$V_f = \left\{ \frac{A_h}{1 - \mu} [m_0 + m_f A_f^\sigma + m_h (1 - (1 - A_h)(V_h/V_f))^\sigma)] \right\}^{\frac{1}{1-\sigma}}$$

It is straightforward to show that V_f and V_h approach 1 as $\rho, \tau \to 0$. By Lemma 1, when ρ is sufficiently small, then $V_f^* > V_h^*$, in which case the conditions in (8)-(9) reduce to:

$$[(1 + A_f)^\sigma - 1]^{\frac{1}{1-\sigma}} < V_f < A_f^{\frac{\sigma}{1-\sigma}}$$
$$[(1 + A_h)^\sigma - 1]^{\frac{1}{1-\sigma}} < V_h < A_h^{\frac{\sigma}{1-\sigma}}$$
$$V_f^\sigma - V_f > V_h^\sigma - V_h \tag{24}$$
$$(A_h V_h + V_f - V_h)^\sigma - V_f > A_h^\sigma V_h^\sigma - V_h$$
$$V_h + V_f^\sigma > (V_f + A_h V_h)^\sigma$$

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[11] This is because the amount of goods produced by a rich seller converges to the quantity produced by a poor seller as A_f and A_h approach 1. So there is nothing to gain by waiting to meet a poor seller.

a set of constraints that mirrors (19)-(23). Given these expressions we can state the following proposition:

Proposition 2. *Consider $u(q) = q^\sigma$. If ρ is sufficiently small then the currency competition equilibrium does not exist.*

The proof is immediate: the third inequality in (24) is violated when V_f^* is close to 1. Comparing Propositions 1 and 2 we find that depending on the functional form of preferences, the currency competition equilibrium may or may not exist for the same parameter values. This seems surprising but is the result of the properties of the net gain from spending a dollar relative to the home currency under these two utility specifications. The key element is whether or not $S(V)$ is monotonically increasing or not. To see this consider $S(V)$ for the two utility specifications. For $u(q) = q^\sigma$, $S(V)$ is decreasing for values of V close to 1. In this case, for ρ small, $V_f^* > V_h^*$ and V_f^* is close to 1. Thus, it must be the case that $S(V_f^*) < S(V_h^*)$, so $p_0 = 0$ is optimal and the currency competition equilibrium cannot exist. However, when $u(q) = q^\sigma + q$, $S(V)$ is monotonically increasing, hence for any $V_f^* > V_h^*$, it must be the case that $S(V_f^*) > S(V_h^*)$ so $p_0 = 1$ is always optimal, which is needed for the currency competition equilibrium to exist.

6   The Gresham's Law Equilibrium: $p^* = (0,0,0)$

We now want to consider a world in which $p^* = (0,0,0)$. In this equilibrium, fh buyers tend to 'hoard' the good (safe) currency and spend the bad (risky) currency.[12] This equilibrium transaction pattern has the flavor of Gresham's Law – the circulation of good money is reduced while the circulation of bad money increases – and so we refer to it as such.

The solution procedure in the case where $d = 1$ and $p^* = (0,0,0)$ follows directly from that

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[12]This is a common occurrence in many developing and transitional economies – agents use dollars for some transactions but carry out a majority of purchases using the risky home currency. Several papers have tried to model this phenomenon [see Chang (1994), Uribe (1997), Engineer (2000), Sibert and Liu (1998)]. The main drawback of these models is that they all rely on an ad-hoc assumption that the foreign currency has a relatively higher 'transaction cost' (or trading friction) associated with its use as a medium of exchange. We want to consider a world in which the fundamental trading environment and all trading frictions are the same for each currency.

above. The value function expressions in (4) are unchanged, (5) becomes

$$V_{fh} = V_h + A_{fh}V_f$$

while the quantities in (6) still hold with the only changes being

$$q_h^f = A_{fh}V_f \ \text{ and } \ q_f^h = V_h - (1 - A_{fh})V_f$$

Substituting the equilibrium quantities into V_f and V_h yields

$$V_f = \frac{A_f\left[m_0 u(V_f) + m_f u(A_{fh}V_f) + m_h u(A_f V_f)\right]}{1 - \mu} \tag{25}$$

$$V_h = \frac{A_h\left[m_0 u(V_h) + m_f u(A_h V_h) + m_h u(V_h - (1 - A_{fh})V_f)\right]}{1 - \mu} \tag{26}$$

where (26) is defined only for $V_h > (1 - A_{fh})V_f$. Once again, in a monetary equilibrium (V_f^*, V_h^*) must be a positive fixed point of the map defined by (25)-(26).

Lemma 2. *Suppose $d = 1$ and $p^* = (0, 0, 0)$ is a dual currency monetary equilibrium. Then there always exists a unique positive V_h^*, and $V_f^* = V_h^*$ whenever currency risk is absent. If currency risk is present, a fixed point $V_f^* > V_h^* > 0$ exists whenever trading frictions are sufficiently limited. This requirement is also sufficient to guarantee that (V_f^*, V_h^*) is unique and that $\frac{V_f^*}{V_h^*} < \frac{1 - A_h}{1 - A_{fh}}$.*

Determining the conditions under which the conjectured buying strategies are individually optimal follows from (7)-(9). With regards to the equilibrium distribution of money holdings, the constraints (1) and (2) still hold but the steady-state flow conditions change and are listed in the appendix. As before, we can generate sufficient conditions for an equilibrium distribution to exist and can show that it is unique.

Note again that under the conjectured trading strategy $V_f = 0$ solves (25). Given $V_f = 0$, there is a unique value of $V_h > 0$ that solves (26). This corresponds to a situation in which dollars do not circulate at all despite being a less risky currency. In this case, only the bad currency circulates. This is the extreme version of Gresham's Law. Again, this type of monetary equilibrium is a simple variation of the mono-currency equilibrium studied by Camera-Corbae with the only difference being that there is a 'tax' on the circulating currency. Again, we ignore this equilibrium and focus

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our attention on the dual-currency equilibrium. It is also important to recognize that $V_f > V_h = 0$ is never an equilibrium, if $p^* = (0, 0, 0)$.

By setting $u(q) = q^\sigma + q$ we can derive a set of conditions comparable to (17)-(23), which are contained in the appendix. As noted earlier a critical requirement now is that $V_f < V_h$ for $p_0 = 0$ to be optimal. This inequality is at the core of the following proposition:

Proposition 3. *Consider* $u(q) = q + q^\sigma$. *If* ρ *is sufficiently small then the Gresham's Law equilibrium does not exist.*

This follows from the fact that limited trading frictions only support $V_f^* > V_h^*$ (by Lemma 2). This, combined with the monotonicity of $S(V)$, makes $p_0 = 1$ optimal. Hence, the Gresham's Law equilibrium cannot exist for ρ small. In order for it to exist, either the parameters yield $V_f^* < V_h^*$, which seems unlikely when the home currency is risky, or $S(V)$ cannot be monotonically increasing in V.

For the utility function $u(q) = q^\sigma$, the following is proved:

Proposition 4. *Consider* $u(q) = q^\sigma$ *and a stationary distribution supporting the transaction pattern* $d = 1$ *and* $p^* = (0, 0, 0)$. *There exist positive values* $\hat{\sigma}_H$ *and* $\hat{\rho}_H$ *such that if* $\sigma \in (0, \hat{\sigma}_H)$ *and* $\rho \in (0, \hat{\rho}_H)$ *then the Gresham's Law equilibrium exists and is unique.*

The reason this equilibrium exists is because the parameter restrictions on ρ and σ ensure that the solutions for V_f^* and V_h^* lie on the decreasing portion of $S(V)$. In this case, even though the dollar is more valuable and buys more goods, it is also a very valuable asset to give up. Hence, the net gain from spending a dollar is very low while the net gain from spending the home currency is higher despite its riskiness.

7   Transaction Patterns and Relative Circulation

Here we address two issues by means of numerical analysis.[13] First, we discuss the existence of equilibria when $d = 1$ for all eight possible pure strategy vectors $p^* = (p_0^*, p_f^*, p_h^*)$. Second, we demonstrate how the currencies' transactions velocity responds to changes in the home currency risk.

To illustrate the importance of trading frictions and home currency risk, for existence of equilibria, we let ρ and τ be free to vary, for the baseline parameterization. When $u(q) = q^\sigma + q$, only $p^* = (1, 1, 1)$ is an equilibrium, while Figure 1 displays the different equilibria existing when $u(q) = q^\sigma$. The figure confirms the intuition developed via the propositions: if the home currency risk is low and the economy is functioning well (trading is relatively easy to accomplish), then home agents will spend the home currency first when conducting internal trades. However, if home currency risk is high and the economy is not functioning well, then currency substitution occurs and agents prefer to spend the foreign currency. In Figure 1, the currency competition equilibrium, $p^* = (1, 1, 1)$, occurs when i) the home currency risk is high and trading frictions are reasonably low, ii) trading frictions are high but τ is low or iii) both are high. The Gresham's Law equilibrium, $p^* = (0, 0, 0)$, occurs when i) trading frictions are high and the home currency risk is very low, ii) trading frictions are very low and the currency risk is sufficiently high, or iii) both are low.

The intuition for this is as follows. When trading frictions are very low, buyers know that they will meet another seller very quickly. If the home currency risk is also relatively low, then prices in terms of the home currency are not much higher than dollar prices. However, by spending the home currency, the fh buyer gets rid of the risky currency. In addition, because trading frictions are low, he does not have to wait too long to spend the dollar because another trading opportunity will arise quickly. Hence, he prefers to dump the risky currency even if he consumes a little less today. When trading frictions are high, the fh buyer knows that he will not consume again for awhile, hence he desires a sufficient amount of consumption today if a trading opportunity arises.

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[13]The results are presented via illustrations that were generated in the following way: 1) conjecture an equilibrium vector p^*, 2) pick a pair of values for the variables defined on the axes of each figure, 3) use this pair of values to solve for the equilibrium distribution and value functions, 4) then check to see if the conjectured strategy vector p^* is individually optimal. If an equilibrium exists, then that parameter pair is shaded. This was done for 1 million such pairs, for each of the 8 pure strategy candidate vectors p^*. In all illustrations (unless otherwise noted) $x = 0.4$, $\sigma = 0.5$, $\rho = 0.08$, $\alpha = 5$, $M_f = .75$, and $M_h = .25$.

This leads him to spend the dollar and hold onto the home currency despite the risk of losing it in the future.

What we would like to know, is how the relative circulation of the two currencies changes in response to changes in home currency risk, τ. In general, circulation is affected by two elements: the sellers' willingness to accept the currency, and the buyers' willingness to spend it. By construction, however, sellers always accept both currencies in our equilibria. Hence, for given supplies of the two currencies, changes in their equilibrium circulation are driven by changes in their distribution and the spending pattern. In order to measure the degree of circulation of each currency, we use the transactions velocity that, we emphasize, is endogenous in our model.

The transaction velocity is the amount traded per unit time, divided by its stock. When $d = 1$ we define velocities as:

$$v_f \propto \alpha x\{(1-\mu)(m_f + m_{2f}) + (p_0 m_0 + p_f m_f + p_h m_h)m_{fh}\}$$
$$v_h \propto \alpha x\{(1-\mu)(m_h + m_{2h}) + [(1-p_0)m_0 + (1-p_f)m_f + (1-p_h)m_h]m_{fh}\}.$$

The first term is the fraction of each currency that changes hands when buyers holding only that currency meet sellers and spend one unit of their holdings. The second term captures how the spending behavior of the buyer with a mixed portfolio affects the relative velocities of each currency. Velocities are affected by the steady-state distribution of money holdings and by the equilibrium strategy vector p^*. In particular, a change in p^* moves v_f and v_h in opposite directions, ceteris paribus.[14] Thus, the government's confiscation/injection policy affects the velocity of each currency via changes in the distribution of money holdings and the buyers' trading strategies.

Figure 2 illustrates the transaction velocities corresponding to the equilibria depicted in Figure 1 when ρ is fixed at its baseline value, and τ is free to vary. Given that there is more home than foreign currency ($M_f = .75$, $M_h = .25$), the transaction velocity for the home currency is always the highest since more trades are being conducted with it, than the foreign currency. When $\tau = 0$, $v_f = .74$, and $v_h = .15$; as the risk on the home currency increases, however, the velocities change as the distribution of money holdings and the transaction pattern change. We can see that, for an equilibrium associated with a given p^*, increases in currency risk lead to small declines in v_h and small increases in v_f. Once the risk gets high enough, buyers with mixed portfolios begin spending the foreign currency, rather than the home. Thus more transactions involve dollars, so that v_h

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[14]Note that if $p^* = (1,1,1)$ and $(m_f + m_{2f}) \approx (m_h + m_{2h})$, then $v_f > v_h$ and vice versa if $p^* = (0,0,0)$.

falls and v_f increases. As the spending pattern changes, there are dramatic decreases in v_h and large increases in v_f. When $\tau = 1$, then $v_f = .55$, $v_h = .28$ and the ratio v_f/v_h rises to .51 (from .20 at $\tau = 0$). These results seem very intuitive and suggest that as the home currency becomes increasingly risky, people 'substitute' out of the bad currency into the good currency causing the circulation of the bad currency to fall and the circulation of the good currency to increase.

We next analyze how varying the degree of home currency risk, τ, and the ratio of the home to the foreign money stock affects the equilibrium transaction pattern by varying the relative supplies of currencies when $M_f + M_h = M = 1$. Figure 3 illustrates the equilibria when $u(q) = q^\sigma$, for the baseline parameterization. Its main feature is that the equilibrium transaction pattern is not driven by the relative amount of home currency in the economy. Rather, home currency risk is the critical parameter. We also observe an interesting spending pattern. Given a value M_h/M, the fh buyer always spends the home currency for low levels of home currency risk. As the risk factor rises, this buyer begins spending the dollar when buying from sellers who already hold a unit of the home currency, i.e. $p^* = (0,0,1)$. This occurs because the h sellers charge a low dollar price in order to acquire a unit of safe currency to diversify their portfolio. As home currency risk continues to increase, the fh buyer also starts spending the dollar on f sellers, i.e. $p^* = (0,1,1)$. Finally, when the home risk is high enough, all sellers charge high prices in terms of the home currency, i.e. $p^* = (1,1,1)$. Hence, buyers with a mixed portfolio always prefer to make dollar purchases.

Executing a similar exercise for $u(q) = q + q^\sigma$ generates only the only equilibrium $p^* = (1,1,1)$. We had to decrease σ to 0.15 and ρ to 0.02 in order to find other equilibria. The results appear in Figure 4.[15] Still, despite the fact that there are eight possible vectors p, only two of them are an equilibrium, and are unique: $p^* = (1,1,1)$ and $p^* = (0,1,1)$. In Figure 4, when the home currency risk is very low, $p^* = (1,1,1)$ is an equilibrium even when dollars form less than half of the available currency. However, as τ rises, $p^* = (1,1,1)$ is an equilibrium only if there is a large supply of dollars. This corresponds to the idea of the economy being 'highly dollarized' - dollars are the dominant source of currency, and the preferred medium of exchange. On the other hand, if only few dollars are present in the economy, then $p^* = (0,1,1)$ is the unique equilibrium. In this situation, agents holding a mixed portfolio only spend the dollar on rich sellers who charge a much higher price for home currency. Poor sellers offer better prices in terms of home currency, since

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[15]Interestingly, if $u(q) = q^\sigma$, $\sigma = 0.15$, and $\rho = 0.02$ then only $p^* = (1,1,1)$ exists.

they need cash; thus the buyer can afford to spend the bad currency in those trades.

8   Conclusion

We have constructed a dual currency search model to study Gresham's Law and currency competition from first principles. We have investigated how changing levels of risk on a home currency affects agents' transaction patterns and thus their willingness to use a safer foreign currency as a preferred medium of exchange. Our results demonstrate that small changes in the degree of home currency risk can result in lower circulation of the risky currency and higher circulation of the safer currency.

Our analysis also contributes to the understanding of some aspects of the phenomenon commonly known as "dollarization", a concept that has a wide variety of meanings and uses. One of the most common and basic forms of dollarization is the simultaneous use of a foreign currency alongside the home currency as a media of exchange. This phenomenon is commonly associated to the concept of currency substitution (Calvo and Végh, 1992).

Given that two currencies are accepted as media of exchange in an economy, it is the extent of the currency substitution taking place that is the relevant issue for policymakers. That is, what is the relative use of the foreign currency to the home? Our theoretical analysis allows us to consider this question by focusing on key determinants in the patterns of circulation of a medium of exchange, namely trading frictions and currency risk.

We find that a poorly functioning economy with risky home currency, is prone to dollarization. Thus our analysis is consistent with the view that home agents will continue using the home currency in internal trade if the purchasing power risk is kept very low, but once that risk gets too high substantial currency competition kicks in. The normative aspect of our results is that a low dollarized economy can avoid becoming highly dollarized by implementing policies aimed at reducing currency risk and improving the trading environment so that the economy functions well. At the same time our results serve as a warning that dollarization will be unavoidable if currency risk is not kept under control.

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Appendix

Existence and uniqueness of a stationary distribution of portfolios when $p^* = (1,1,1)$.

I. Sufficient conditions for existence. We use a procedure similar to that used by Zhou (1997). Specifically, consider the state space M and an equilibrium point $m^* \in M$. Define a real-valued function L on M, that satisfy the following requirements: (i) L is continuous and has continuous first-partial derivatives (ii) $L(m)$ has a unique minimum at m^* with respect to all other points in M. (iii) The function $\dot{L}(m)$ satisfies $\dot{L}(m) \leq 0$ for all $m \in M$. This function L is called a Liapunov function. We then rely on the Liapunov theorem stating that if there exists a Liapunov function the equilibrium point m^* is stable and if the function $\dot{L}(m) < 0$ at all $m \neq m^*$ then the stability is asymptotic.

Equations (1)-(2) imply that $\{m_0, m_f, m_h, \eta\}$ are single-valued functions of $\{m_{2f}, m_{2h}, mf_h\}$:

$$m_f = M_f - m_{fh} - 2m_{2f}$$
$$m_h = M_h - m_{fh} - 2m_{2h} \tag{27}$$
$$m_0 = 1 - M_f - M_h + m_{2f} + m_{2h} + m_{fh}$$

and the government budget constraint

$$\eta = \frac{\tau M_h}{1 - (m_{2f} + m_{2h} + m_{fh})} \tag{28}$$

Using (27) in (12)-(14) we get:

$$\dot{m}_{2f} = (M_f - m_{fh} - 2m_{2f})(M_f - 2m_{2f}) - m_{2f}(1 - M_f + m_{2f} - m_{2h})$$
$$\dot{m}_{2h} = x\left[(M_h - m_{fh} - 2m_{2h})^2 - m_{2h}(1 - M_h - m_{2f} + m_{2h})\right]$$
$$+\eta(M_h - m_{fh} - 2m_{2h}) - \tau m_{2h} \tag{29}$$
$$\dot{m}_{fh} = x[(M_f - m_{fh} - 2m_{2f})m_{2h} + (M_h - m_{fh} - 2m_{2h})m_{2f}$$
$$+2(M_h - m_{fh} - 2m_{2h})(M_f - m_{fh} - 2m_{2f}) - m_{fh}(1 - M_h - m_{2f} + m_{2h})]$$
$$+\eta(M_f - m_{fh} - 2m_{2f}) - \tau m_{fh}$$

Define the 3x1 vector $m = [m_1, m_2, m_3]$ where $m_1 = m_{2f}$, $m_2 = m_{2h}$, $m_3 = m_{fh}$ and $m_i \in [0,1]$ with $m_1 + m_2 + m_3 \leq 1$. Then define the system in (29) as $\dot{m} = F(m)$ where $F(m)$ is a 3×1 vector. Denote by $F(m)[i]$ the ith row of $F(m)$. Then, letting $\frac{dF(m)[i]}{dm_j} = a(i,j)$, $j, i = 1, 2, 3$, the Jacobian

of $F(m)$ is a 3x3 matrix

$$\frac{dF(m)}{dm} = \begin{bmatrix} a(1,1) & ... & a(1,3) \\ : & a(2,2) & : \\ a(3,1) & ... & a(3,3) \end{bmatrix}$$

where, recalling that τ is a constant,

$$a(1,1) = m_{2h} + 2m_{fh} + 6m_{2f} - 3M_f - 1 < 0$$

$$a(1,2) = m_{2f}$$

$$a(1,3) = 2m_{2f} - M_f < 0 \quad (\text{since } M_f > 2m_{2f})$$

$$a(2,1) = xm_{2h} + \tfrac{d\eta}{dm_{2f}}(M_h - m_{fh} - 2m_{2h}) > 0$$

$$a(2,2) = x\left[4m_{fh} + m_{2f} + 6m_{2h} - 3M_h - 1\right] + \tfrac{d\eta}{dm_{2h}}(M_h - m_{fh} - 2m_{2h}) - 2\eta - \tau$$

$$a(2,3) = -2x\left[M_h - m_{fh} - 2m_{2h}\right] + \tfrac{d\eta}{dm_{fh}}(M_h - m_{fh} - 2m_{2h}) - \eta$$

$$a(3,1) = x\left[4m_{fh} + 4m_{2h} - 3M_h\right] + \tfrac{d\eta}{dm_{2f}}(M_h - m_{fh} - 2m_{2f}) - 2\eta$$

$$a(3,2) = x\left[2m_{fh} + 4m_{2f} - 3M_f\right] + \tfrac{d\eta}{dm_{2h}}(M_f - m_{fh} - 2m_{2f})$$

$$a(3,3) = x\left[4m_{fh} + 4m_{2f} + 2m_{2h} - M_h - 2M_f - 1\right] + \tfrac{d\eta}{dm_{fh}}(M_f - m_{fh} - 2m_{2f}) - \eta - \tau$$

where we note that $\frac{d\eta}{dm_i} > 0$ for all m_i. We note that $M_f > m_{fh} + 2m_{2f}$ and $M_h > m_{fh} + 2m_{2h}$ if $m_f, m_h > 0$, using (27)-(??). Substituting the infimum $M_f = m_{fh} + 2m_{2f}$ and $M_h = m_{fh} + 2m_{2h}$ in $a(2,2), a(3,2)$ and $a(3,3)$, it is easy to show that all of these terms are strictly negative as $\eta \to 0$ while $a(2,3) \to 0^-$. Thus there are small values of $\eta > 0$ such that $a(2,2), a(2,3), a(3,2)$ and $a(3,3)$ are all negative. Note that $\eta \to 0$ when either $\tau \to 0$ or $M_h \to 0$, and so does $\frac{d\eta}{dm_i} > 0$ for all m_i.

We want to show that $\frac{dF(m)}{dm}$ is negative definite. To do so we can consider the sign of its three principal minors:

$$D_1 = a(1,1), \quad D_2 = \begin{vmatrix} a(1,1) & a(1,2) \\ a(2,1) & a(2,2) \end{vmatrix}, \text{ and } D_3 = \begin{vmatrix} a(1,1) & a(1,2) & a(1,3) \\ a(2,1) & a(2,2) & a(2,3) \\ a(3,1) & a(3,2) & a(3,3) \end{vmatrix}$$

We note that $D_1 = a(1,1) < 0$. This is so because $M_f \geq m_{fh} + 2m_{2f}$ (with strict inequality if $m_f > 0$) using (27). Substituting $M_f = m_{fh} + 2m_{2f}$ in $a(1,1)$ provides a maximum for $a(1,1)$. This maximum is seen to be negative since $-m_{fh} + m_{2h} - 1 < 0$.

The minor $D_2 = a(1,1)a(2,2) - a(1,2)a(2,1)$. Note that $a(1,2)$ and $a(2,1)$ are both positive, and that their product tends to zero as x and η shrink to 0. Furthermore, $a(2,2) < 0$ as η tends

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to zero because $-3M_h - 1 + 4m_{fh} + m_{2f} + 6m_{2h} < 0$ (since $M_h \geq m_{fh} + 2m_{2h}$). Thus $D_2 > 0$ for x and η small (i.e. either τ or M_h small).

The third minor is

$$D_3 = a(1,1) \left[a(2,2)a(3,3) - a(2,3)a(3,2) \right]$$
$$-a(1,2) \left[a(2,1)a(3,3) - a(2,3)a(3,1) \right]$$
$$+a(1,3) \left[a(2,1)a(3,2) - a(2,2)a(3,1) \right]$$

Note that as $\eta, x \to 0$ then the second and third line in D_3 vanish, and that the first line, is strictly negative and given by

$$\tau^2 \left(-3M_f - 1 + m_{2h} + 2m_{fh} + 6m_{2f} \right)$$

We conclude that there exist an M_h and x positive but sufficiently small such that $D_1 < 0$, $D_2 > 0$ and $D_3 < 0$. Thus, for M_h and x small the matrix $\frac{dF(m)}{dm}$ is negative definite (see Chiang).

Since $F(m)$ is a 3x1 vector ("'" transposes it), define the function

$$L(m) = \left[F(m) \right]' F(m) = (\dot{m}_{2f})^2 + (\dot{m}_{2h})^2 + (\dot{m}_{fh})^2 \geq 0$$

We show it is a Liapunov function. It is continuous (by construction) and it has continuous first partial derivatives. Recalling that the vector $F(m) = \dot{m}$, that $d[F(m)]'/dt = \dot{m}' \frac{dF(m)}{dm}$ (a 1x3 vector) and that $dF(m)/dt = \left[\frac{dF(m)}{dm} \right]' \dot{m}$ (a 3x1 vector) then the time derivative of $L(m)$ is the quadratic form (a scalar)

$$\dot{L}(m) = \dot{m}' \frac{dF(m)}{dm} \dot{m} + \dot{m}' \left[\frac{dF(m)}{dm} \right]' \dot{m}$$

so that $\dot{L}(m) = 0$ if $\dot{m} = 0$, and < 0 if $\dot{m} \neq 0$ for x and M_h small, since $\frac{dF(m)}{dm}$ is negative definite.

To show that there exists an m^* such that $L(m^*) = 0$ we use a proof by contradiction. If $L(m) = \dot{m} \neq 0$ for all m defined above then $\dot{L}(m) \neq 0$. Since m is defined on a compact set it follows that $\dot{L}(m)$ has a maximum, say $l < 0$ (because of negative definiteness). But this cannot be since, defining $m(t)$ to be the state of the system at date t,

$$\int_0^t \dot{L}(m(s))ds = L(m(t)) - L(m(0)) \leq lt \Rightarrow L(m(t)) \leq lt + L(m(0))$$

which in turn implies $L(m(t)) \to -\infty$ as $t \to \infty$. This can't be since at every date, by construction, $L(m) \geq 0$. Thus $L(m)$ must be reaching a minimum 0 at some m^*. To show that m^* is unique, see below.

Thus $L(m)$ is a Liapunov function, and applying the Liapunov Theorem (see Azariadis, 1993, for a discrete time version) the unique equilibrium m^* is asymptotically stable if x and M_h are positive but sufficiently small. The money distribution m^* is unique and stationary.

II. Uniqueness. Using (27)-(28) and $M_f + M_h < 2$, then $m_i > 0$ and $\eta < 1$ require

$$m_{fh} + 2m_{2f} < M_f < 2 - M_h < 2 - m_{fh} - 2m_{2h}$$

$$m_{fh} + 2m_{2h} < M_h < \frac{1 - (m_{2f} + m_{2h} + m_{fh})}{\tau} \quad .$$

We now show that for a feasible pair $\{m_{2h}, m_{2f}\}$, if m_{fh}^* solves (14), then it must be unique. Using (14) and (a1)-(a4) we obtain

$$m_{fh} = \frac{(M_f - m_{fh} - 2m_{2f})\left(\frac{\tau M_h}{1 - m_{2f} - m_{2h} - m_{fh}} + x m_{2h}\right)}{\tau + x\left(1 - M_h + m_{2h} - m_{2f}\right)}$$
$$+ \frac{x\left(M_h - m_{fh} - 2m_{2h}\right)\left[m_{2f} + 2\left(M_f - m_{fh} - 2m_{2f}\right)\right]}{\tau + x\left(1 - M_h + m_{2h} - m_{2f}\right)}.$$

The right hand side can be shown to be strictly decreasing in m_{fh} for all feasible values of m_{fh}, m_{2h}, and m_{2f}. It then follows that if there is a feasible m_{fh}^* that solves this expression, then it is unique.

We now show that for a feasible value of m_{fh}, a unique pair $\left\{m_{2h}^*, m_{2f}^*\right\}$ solves (12) and (13). Using (12) and (a1)-(a4) we obtain $m_{2h} = f(m_{fh}, m_{2f})$ where

$$f(m_{fh}, m_{2f}) = 1 - M_f + m_{2f} - \frac{(M_f - m_{fh} - 2m_{2f})(M_f - 2m_{2f})}{m_{2f}}$$

which is easily seen to be increasing in m_{2f} for feasible values $m_{2f} \leq (M_f - m_{fh})/2$. Furthermore it is concave in m_{2f}.

Using (13) and (a1)-(a4) we obtain $m_{2f} = h(m_{fh}, m_{2h})$ where

$$h(m_{fh}, m_{2h}) = \frac{\tau}{x} + 1 - M_h + m_{2h} - \frac{[\eta + x\left(M_h - m_{fh} - 2m_{2h}\right)]\left(M_h - m_{fh} - 2m_{2h}\right)}{x m_{2h}}$$

which is easily seen to be increasing and concave in m_{2h} for feasible values $m_{2h} \leq (M_h - m_{fh})/2$, since η is increasing in m_{2h}. Note also that $f(m_{fh}, m_{2f}) \to -\infty$ as $m_{2f} \to 0$ and $h(m_{fh}, m_{2h}) \to -\infty$ as $m_{2h} \to 0$. Note that $m_{2h} \leq (M_h - m_{fh})/2 < f(m_{fh}, (M_f - m_{fh})/2)$ and $m_{2f} \leq (M_f - m_{fh})/2 < h(m_{fh}, (M_h - m_{fh})/2)$. The properties of the two functions imply there is a single crossing point for the two functions in the feasible part of the (m_{2h}, m_{2f}) plane. Thus, for any feasible value of

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m_{fh} and η, there is a unique pair $\left\{m_{2h}^*, m_{2f}^*\right\}$ that solves the system

$$\begin{cases} m_{2h} = f(m_{fh}, m_{2f}) \\ m_{2f} = h(m_{fh}, m_{2h}) \end{cases}$$

Given the uniqueness of the values in (27)-(28) and m_{fh}, then if a feasible distribution exists, it is unique.

Proof of Lemma 1.

Conjecture $d = 1$ and $p^* = (1, 1, 1)$ and consider $\tau > 0$. Using (10) it follows that

$$\rho V_h = x\left[m_0 u(V_h) + m_f u(A_h V_h) + m_h u(A_h V_h)\right] - x(1 - \mu)V_h - \tau V_h \equiv H(V_h)$$

$H(V_h)$ is a monotonically increasing and strictly concave function of V_h, which starts at 0 and has a decreasing first derivative that vanishes as $V_h \to \infty$. Thus, it has two fixed points, one is $V_h = 0$ (the mono-currency equilibrium, which we ignore), and the other is $V_h^* > 0$.

Given V_h^* use once again (3) and (11) so that in equilibrium

$$\rho V_f = x\left[m_0 u(V_f) + m_f u(V_f - (1 - A_h)V_h^*) + m_h u(A_f V_f)\right] - x(1 - \mu)V_f \equiv F\left(V_f, V_h^*\right)$$

defined only for $V_f \geq (1 - A_h)V_h^*$. $F\left(V_f, V_h^*\right)$ is strictly concave and monotonically increasing in V_f. As $V_f \to^+ (1 - A_h)V_h^*$ then $F\left(V_f, V_h^*\right)$ converges to a positive value, and its slope becomes unbounded. Thus, the intermediate value theorem suggest there can be at most two positive fixed points to the map $\rho V_f = F\left(V_f, V_h^*\right)$.

1. A fixed point $V_f^* > V_h^*$ exists if

$$\rho V_f - F\left(V_f, V_h^*\right)|_{V_f = V_h^*} < 0 \quad \Leftrightarrow \quad H\left(V_h^*\right) < F\left(V_h^*\right) \tag{30}$$

since $F(V_f, V_h^*)$ is strictly concave and V_h^* satisfies $\rho V_h^* = H\left(V_h^*\right)$. Using the definition of $H\left(V_h^*\right)$, rearrange (30) as

$$H\left(V_h^*\right) - F\left(V_h^*\right) = -x m_f\left[u(A_f V_h^*) - u(A_h V_h^*)\right] - \tau V_h^* < 0 \tag{31}$$

always satisfied since $A_h < A_f$. Hence $V_f^* > V_h^*$ always exists when $p^* = (1, 1, 1)$ is an equilibrium and $\tau > 0$.

2. Notice that (31) holds as an equality iff $\tau = 0$, so that $V_f^* = V_h^*$ is the unique positive fixed point.

3. If $V_f^* > V_h^*$, then $\frac{V_f^*}{V_h^*} \leq \frac{1-A_h}{1-A_f}$ for all $\tau \geq 0$ if ρ is small This is so whenever

$$\left. \frac{F(V_f, V_h^*)}{H(V_h^*)} \right|_{V_f=V_h^*} \equiv \frac{m_0 u(V_h^*) + m_f u(A_h V_h^*) + m_h u(A_f V_h^*) - (1-\mu)V_h^*}{m_0 u(V_h^*) + m_f u(A_h V_h^*) + m_h u(A_h V_h^*) - (1-\mu)V_h^* - (\tau/x)V_h^*}$$

$$\leq \frac{1-A_h}{1-A_f}$$

satisfied with equality as $\tau \to 0$ since $F(V_f, V_h^*) \to^+ H(V_h^*)$; if $\tau > 0$ then $\lim_{\rho \to 0} \frac{1-A_h}{1-A_f} \to \infty$ but $F(V_h^*)/H(V_h^*)$ is bounded. Concavity of $F(V_f, V_h^*)$ completes the argument.

4. We now show that if another fixed point V_f^{**} of the map $\rho V_f = F(V_f, V_h^*)$ exists when $\tau > 0$, it must be such that $V_f^{**} < V_h^*$. To show it note that $F(V, V_h^*) = 0$ for some positive $V = V^L < (1-A_h)V_h^*$. However, $H(V^L) > 0$ since it is increasing, and $H(0) = 0$. That is $\exists \, 0 < V^L < (1-A_h)V_h^*$ such that $F(V^L, V_h^*) = 0 < H(V^L)$. Since $V_f^* > V_h^*$ always exists, then it must be that $F(V, V_h^*)$ intersects $H(V)$ at some point $V^H \geq (1-A_h)V_h^*$, i.e. $F(V^H, V_h^*) = H(V^H)$, satisfied iff

$$xm_h \left[u(V^H - (1-A_h)V_h^*) - u(A_h V^H) \right] + xm_f \left[u(A_f V^H) - u(A_h V^H) \right] + \tau V^H = 0$$

Since $A_f > A_h$ then this last equality can be satisfied only if $u(V^H - (1-A_h)V_h^*) - u(A_h V^H) < 0$, i.e. if $V^H < V_h^*$. Since $F(V^H, V_h^*) = H(V^H) \geq \rho V^H$ (i.e. the functions intersect above the line traced by ρV) it must be that $V_f^{**} < V^H < V_h^*$.

5. $V_f^{**} < V_h^*$ cannot exist if

$$\rho V_f - F(V_f, V_h^*)|_{V_f=(1-A_h)V_h^*} < 0 \quad \Leftrightarrow \quad (1-A_h)H(V_h^*) < F((1-A_h)V_h^*, V_h^*) \tag{32}$$

which we can rewrite as

$$m_0 u((1-A_h)V_h^*) + m_f u(A_f(1-A_h)V_h^*) - (1-A_h)[m_0 u(V_h^*) + m_f u(A_h V_h^*)]$$
$$> (1-A_h)\left[m_h u(A_h V_h^*) - \tfrac{\tau}{x}V_h^* \right] \tag{33}$$

The LHS of the inequality is always positive $\forall \tau > 0$ since $A_f > A_h$ and $u((1-A_h)k) > (1-A_h)u(k)$ for any $k > 0$, due to Jensen's inequality. The RHS of the inequality is negative

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if τ is close to one since $V_h^* > x m_h u(A_h V_h^*)$; it is also decreasing in τ since A_h and V_h^* increase as τ shrinks. Hence by the intermediate value theorem (33) holds for any ρ if τ sufficiently large. However, it can be shown that (33) holds for τ if ρ is sufficiently small. To do so note that as $\rho \to 0$ then $LHS, RHS \to 0$, but $LHS > RHS$ in the limit since the LHS has a partial relative to ρ, compared to RHS, when ρ is around 0.

6. If $d = 1$ and $p^* = (1, 1, 1)$ is an equilibrium, then equilibria with $V_f^{**} < V_h^*$ and $V_f^* > V_h^*$ cannot coexist, i.e. they are mutually exclusive. To prove it consider the first constraint in (9), $S(V_f) > S(V_h)$. If $S(V)$ is monotonically increasing then $S(V_f) > S(V_h)$ only if $V_f > V_h$ (never if $V_f < V_h$). If $S(V)$ is hump-shaped then (i) if V_h is on the decreasing segment of $S(V_h)$ then $S(V_f) > S(V_h)$ only if $V_f < V_h$ (never if $V_f > V_h$) and (ii) if V_h is on the increasing segment of $S(V_h)$ then $S(V_f) > S(V_h)$ only if $V_f > V_h$ (never if $V_f < V_h$). ∎

Proof of Proposition 1.

Consider an equilibrium distribution that satisfies (1)-(2) and (12)-(14). From a prior discussion we know that it exists, under certain conditions.

It is straightforward to show that (22) and (23) are satisfied as strict inequalities, whenever $1 < \frac{V_f}{V_h} \leq \frac{1-A_h}{1-A_f}$. By continuity $\frac{1-A_h}{1-A_f} < \frac{V_f}{V_h}$ also satisfies these two inequalities if $\frac{V_f}{V_h}$ is above–but close to–$\frac{1-A_h}{1-A_f}$. From Lemma 1 we know that there always exists a unique fixed point of (15)-(16) $V_f^* > V_h^*$ when $\tau > 0$, such that $V_f^*/V_h^* \leq \frac{1-A_h}{1-A_f}$ for $\rho > 0$ small. As a result, we know that there exists a $\rho_{H1} > 0$ such that (22) and (23) and Lemma 1 are satisfied for some $\rho \in (0, \rho_{H1})$.

What remains to be shown is that when $V_f^* > V_h^*$ satisfies (17) and (18), then it also satisfies (19) and (20). The intervals defined by the bounds in (19) and (20) are non-empty for all values of A_f, A_h and σ. Furthermore, A_f and A_h converge to 1 as ρ approaches zero. Comparing the expressions in (17) and (18) to the respective upper bounds in (19) and (20) it is easy to verify the existence of a value of positive $\rho_H < \rho_{H1}$ such that for $\rho \in (0, \rho_H)$, (17) is below the upper bound in (20) and (18) is below the upper bound in (19) for all $V_f^* > V_h^*$. Furthermore, as σ approaches zero, the lower bounds of (19) and (20) approach zero, while (17) and (18) converge to positive values. Consequently, there exists a $\sigma_H \in (0, 1)$ such that if $\sigma \in (0, \sigma_H)$, and $\rho \in (0, \rho_H)$, then there is always a unique positive fixed point $V_f^* > V_h^*$ that satisfies (17) and (18), and it also

satisfies (19)-(23), i.e. an equilibrium exists such that the conjectured transaction pattern $d = 1$ and $p^* = (1, 1, 1)$ is individually optimal.∎

Proof of Lemma 2.

Conjecture $d = 1$ and $p^* = (0, 0, 0)$ and consider $\tau > 0$. Inspection of (25) shows that its RHS is a monotonically increasing strictly concave function of V_f, which starts at 0 and has a decreasing first derivative that vanishes as $V_f \to \infty$. Thus, it has two fixed points, one is $V_f = 0$ (the mono-currency equilibrium, which we ignore), and the other is $V_f^* > 0$ which satisfies

$$(1 - \mu) V_f = A_f \left[m_0 u(V_f) + m_f u(A_{fh} V_f) + m_h u(A_f V_f) \right] \tag{34}$$

Given V_h^*, Consider the map defined by (26) i.e.

$$V_h = \frac{A_h \left[m_0 u(V_h) + m_f u(A_h V_h) + m_h u(V_h - (1 - A_{fh}) V_f^*) \right]}{1 - \mu} \equiv H\left(V_h, V_f^*\right)$$

defined only for $V_f > (1 - A_{fh}) V_f^*$. The function $H\left(V_h, V_f^*\right)$ is strictly concave for $V_h \geq (1 - A_{fh}) V_f^*$, and monotonically increasing in V_h. As $V_h \to^+ (1 - A_{fh}) V_f^*$ then $H\left(V_h, V_f^*\right)$ converges to a positive quantity, and its slope becomes unbounded. Thus, the intermediate value theorem suggest there can be two positive fixed points to the map $V_h = H\left(V_h, V_f^*\right)$. A fixed point such that $V_f^* > V_h^*$ exists if

$$V_h - H\left(V_h, V_f^*\right)\big|_{V_h = V_f^*} > 0 \quad \Leftrightarrow \quad V_f^* > F\left(V_f^*\right) \tag{35}$$

Furthermore, it will be unique if

$$V_h - H\left(V_h, V_f^*\right)\big|_{V_h = (1 - A_{fh}) V_f^*} < 0 \quad \Leftrightarrow \quad (1 - A_{fh}) V_f^* < H\left((1 - A_{fh}) V_f^*, V_f^*\right) \tag{36}$$

Using (34) we can rewrite the inequality in (35) as

$$\begin{aligned} &A_f \left[m_0 u(V_f^*) + m_f u(A_{fh} V_f^*) + m_h u(A_f V_f^*) \right] \\ &> A_h \left[m_0 u(V_f^*) + m_f u(A_h V_f^*) + m_h u(A_{fh} V_f^*) \right] \end{aligned} \tag{37}$$

Let V_f^* be any positive constant. Recall that $A_{fh} > A_f > A_h$. It follows that as $\rho \to \infty$ then $A_{fh}, A_h, A_h \to 0$, hence both sides of the inequality converge to zero. As $\rho \to 0$ then $A_{fh}, A_h \to 1$ but $A_h < 1$, and the RHS side of the inequality converges to a positive number smaller than the

LHS. It is easy to show that both sides of the inequality are decreasing in ρ. Since (37) is satisfied as $\rho \to 0$ small, by continuity there is a $\rho_{H3} > 0$ such that the inequality above holds $\forall \rho \in (0, \rho_{H3})$, in which case $V_f^* > V_h^*$ when $p^* = (1, 1, 1)$, $d = 1$ and $\tau > 0$.

It is a matter of algebra to show that (37) is likely to be violated if $\tau > 0$ small, $x \cong 0$, $m_f \cong 0$, and ρ large. That is, when the trading frictions are large but the risk on the home currency is quite limited. It is also obvious that if $\tau = 0$ then (35) holds as an equality hence $V_f^* = V_h^*$ is the unique positive fixed point (the other fixed point is $V_f = V_h = 0$).

To show that the positive fixed point $\left(V_f^*, V_h^* \right)$ is unique, rewrite inequality (36) as

$$(1 - A_{fh}) A_f \left[m_0 u(V_f^*) + m_f u(A_{fh} V_f^*) + m_h u(A_f V_f^*) \right]$$
$$< A_h \left[m_0 u((1 - A_{fh}) V_f^*) + m_f u(A_h (1 - A_{fh}) V_f^*) \right]$$

Note that as $\rho \to 0$ then $A_{fh} \to 1$ hence both sides of the inequality converge to zero. Let V_f^* be any positive constant. Take the partial of each side of the inequality with respect to ρ, and then take the limit as $\rho \to 0$. In this way, the partial of *RHS* of the inequality is seen to be positive and unbounded since $u' \left((1 - A_{fh}) V_f^* \right) \to \infty$ as $A_{fh} \to 1$. The partial of *LHS* of the inequality, however, is bounded. By the intermediate value theorem it follows that there is a $\rho_{H4} > 0$ such that (32) holds $\forall \rho \in (0, \rho_{H4})$. Hence the equilibrium $\left(V_f^*, V_h^* \right)$ is unique and such that $V_f^* > V_h^*$ given $d = 1$ and $p^* = (0, 0, 0)$.

For ρ small and $\tau > 0$, $\frac{1-A_h}{1-A_{fh}} > \frac{V_f}{V_h}$ always since the left-hand side converges to infinity while the right-hand side converges to a finite number. ∎

Proof of Proposition 3.

If $p^* = (0, 0, 0)$ and $d = 1$ then the laws of motion must satisfy

$$\dot{m}_{2f} = m_f m_f - m_{2f} (m_0 + m_h)$$
$$\dot{m}_{2h} = x[m_h^2 + m_h m_{fh} - m_{2h} (m_0 + m_f)] + \eta m_h - \tau m_{2h}$$
$$\dot{m}_{fh} = x[m_f m_{2h} + m_h m_{2f} + 2 m_h m_f - m_{fh} (m_0 + m_h)] + \eta m_f - \tau m_{fh}$$

and we can apply the same procedure as before to show that a unique stationary distribution exists.

When $u(q) = q^\sigma + q$, we obtain the following expressions for the value functions and the

optimality constraints

$$V_f = \left\{ \frac{A_f\left[m_0 + m_f A_{fh}^\sigma + m_h A_f^\sigma\right]}{1 - \mu - A_f\left[m_0 + m_f A_{fh} + m_h A_f\right]} \right\}^{\frac{1}{1-\sigma}}$$

$$V_h = \left\{ \frac{A_h\left[m_0 + m_f A_h^\sigma + m_h(1 - (1 - A_{fh})/z)^\sigma\right]}{1 - \mu - A_h\left[m_0 + m_f A_h + m_h(1 - (1 - A_{fh})/z)\right]} \right\}^{\frac{1}{1-\sigma}}$$

$$\left[\frac{(1 + A_f)^\sigma - 1}{1 - A_f}\right]^{\frac{1}{1-\sigma}} < V_f < \left(\frac{A_f^\sigma}{1 - A_f}\right)^{\frac{1}{1-\sigma}} \tag{38}$$

$$\left[\frac{(1 + A_h)^\sigma - 1}{1 - A_h}\right]^{\frac{1}{1-\sigma}} < V_h < \left(\frac{A_h^\sigma}{1 - A_h}\right)^{\frac{1}{1-\sigma}} \tag{39}$$

$$V_f < V_h \tag{40}$$

$$A_2^\sigma V_h^\sigma + (1 - A_{fh}) V_f > (1 - A_h) V_h + A_3^\sigma V_f^\sigma \tag{41}$$

$$(1 - A_{fh}) V_f + V_h^\sigma > (V_h + A_{fh} V_f)^\sigma \tag{42}$$

As before, inequalities (38) and (39) on the value functions are needed to ensure that 'rich' buyers only spend one unit of currency and 'poor' buyers buy from 'rich' sellers. Inequalities (40) and (41) are the conditions needed to ensure that the $p^* = (0, 0, 0)$ strategy is optimal. The last inequality (42) ensures the fh buyer only spends the home currency and not both. The surprising feature of these constraints is that despite its riskiness, the home currency must be more valuable than the dollar for this equilibrium to exist. This sharp relationship regarding the magnitude of V_f relative to V_h is a result of the $u(q) = q + q^\sigma$ preference specification.

Note, that (38) and (39) are identical to (17) and (18). As note in the proof of Proposition 1, it follows that they can hold if ρ is sufficiently small. Contradicting this requirement, Lemma 2 has shown that (40) is violated whenever ρ is sufficiently small. It follows that $p^* = (0, 0, 0)$ and $d = 1$ cannot be an equilibrium if trading frictions are too low.∎

Proof of Proposition 4.

The value functions must solve

$$V_f = \left\{ \frac{A_h}{1 - \mu} \left[m_0 + m_f A_{fh}^\sigma + m_h A_f^\sigma\right] \right\}^{\frac{1}{1-\sigma}}$$

$$V_h = \left\{ \frac{A_h}{1 - \mu} [m_0 + m_f A_h^\sigma + m_h(1 - (1 - A_{fh})(V_f/V_h))^\sigma] \right\}^{\frac{1}{1-\sigma}}$$

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It is straightforward to show that V_f^* and V_h^* approach 1 as $\rho, \tau \to 0$. By Lemma 2, when ρ is sufficiently small, it must be that case that $V_f^* > V_h^*$. Consequently, the conditions in (8)-(9) reduce to:

$$[(1 + A_f)^\sigma - 1]^{\frac{1}{1-\sigma}} < V_f < A_f^{\frac{\sigma}{1-\sigma}}$$

$$[(1 + A_h)^\sigma - 1]^{\frac{1}{1-\sigma}} < V_h < A_h^{\frac{\sigma}{1-\sigma}}$$

$$V_f^\sigma - V_f < V_h^\sigma - V_h$$

$$V_f + V_h^\sigma > (V_h + A_{fh}V_f)^\sigma$$

$$(A_f V_f)^\sigma - V_f < (A_{fh}V_f - V_f + V_h)^\sigma - V_h \quad \text{if } \frac{1 - A_h}{1 - A_{fh}} < \frac{V_f}{V_h} \quad \text{(ignore otherwise)}$$

$$(A_{fh}V_f)^\sigma - V_f < (A_h V_h)^\sigma - V_h$$

As before the first four conditions are satisfied when ρ and σ are sufficiently small since A_{fh} and V_f approach 1 while A_h and V_h converge to values less than one for $\tau > 0$. For ρ small and $\tau > 0$, $\frac{1 - A_h}{1 - A_{fh}} > \frac{V_f}{V_h}$ always (see Lemma 2). Finally, for σ close to zero, the last inequality is always satisfied when $\hat{V}_f > V_h$. By the intermediate value theorem we conclude that there exist positive values $\hat{\sigma}_H$ and $\hat{\rho}_H$ such that if $\sigma \in (0, \hat{\sigma}_H)$ and $\rho \in (0, \hat{\rho}_H)$ then a the Gresham's Law equilibrium exists and is unique. ∎
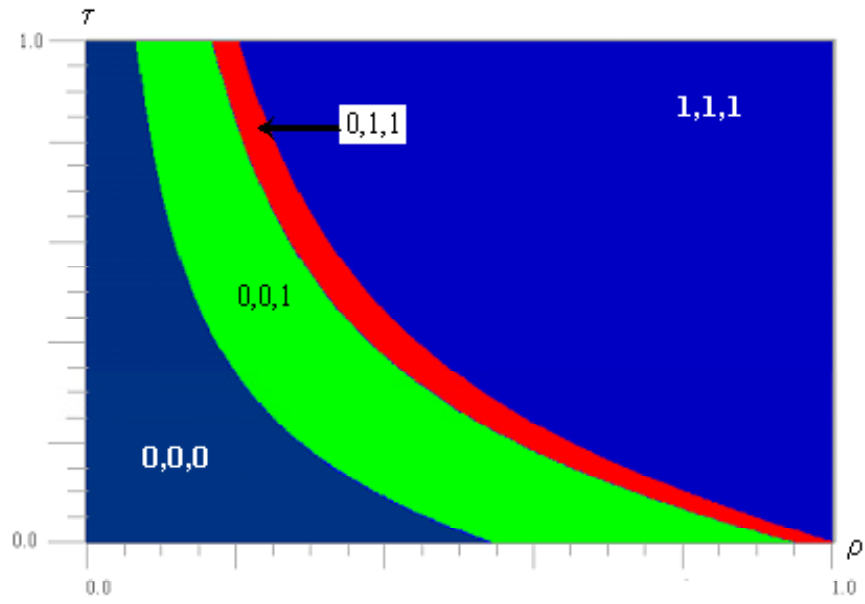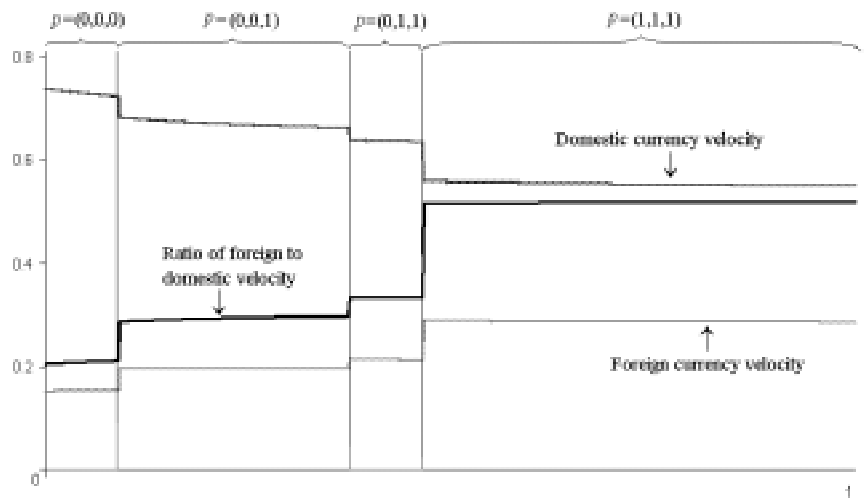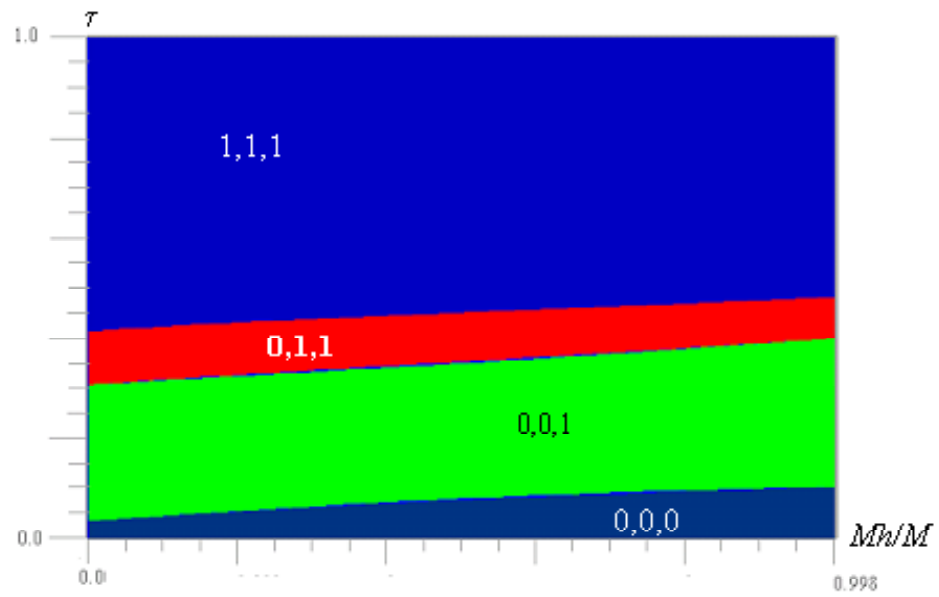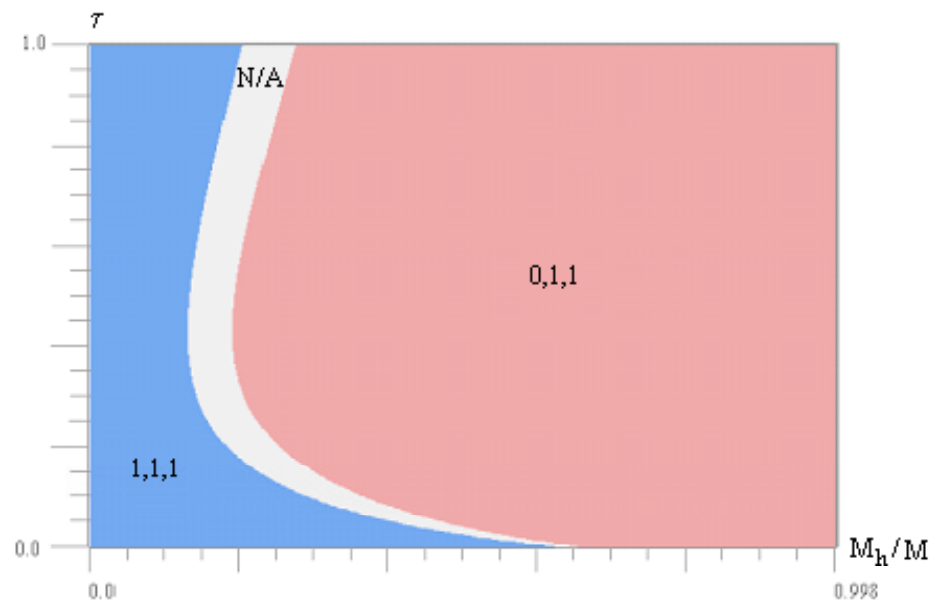
Figure 1

Figure 2

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Figure 3

Figure 4