# Lower bound uncertainty and long-term interest rates

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# Legal Issues

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**Abstract**

Nominal interest rates are constrained by an effective lower bound, but the level of the lower bound
is uncertain. This paper uses a simple shadow rate term structure model to study how lower bound
uncertainty affects long-term interest rates. The main result is that a decline in lower bound uncer-
tainty, in the sense of a mean-preserving contraction of the lower bound distribution, is associated
with a drop in expected future short rates. The effect on the variance of future short rates, and hence
the term premium, is ambiguous. A calibration to Canadian data suggests that a decline in lower
bound uncertainty is associated with a modest drop in long-term interest rates.

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# 1 Introduction

Macroeconomic models and empirical term structure models typically treat the effective lower bound on nominal interest rates as a known parameter. In practice, however, the level of the lower bound is uncertain, and depends on at least three sets of factors. First, it depends on the costs of storing, transporting, and insuring large cash holdings. Second, it depends on the extent to which very low policy rates create unwelcome financial market frictions, and the resulting reluctance of central banks to further lower interest rates.[1] And third, the lower bound is also determined by the level of the "reversal interest rate" (Brunnermeier and Koby, 2018), i.e., the interest rate below which further rate cuts are contractionary, rather than expansionary, because banks stop lending. All these factors are uncertain. The lower bound differs across countries, as the institutional features of financial systems differ. For example, while central banks in the euro area, Denmark, Japan, Norway, Sweden and Switzerland have implemented negative policy rates for prolonged periods – in some cases as low as $-0.75\%$ –, some other major central banks have suggested that the lower bound is zero.[2]

As policy rates in many countries remain low for an extended period of time, market participants may learn more about the structural factors that determine the level of the lower bound, and hence lower bound uncertainty may decline. For example, low interest rates provide an incentive for market participants to inform themselves about the costs of storing cash, which determine the critical rate below which cash is preferable to deposits. Another potential driver of lower bound uncertainty is central bank communication. The effective lower bound is in part determined by how far the central bank is willing to lower its policy rate. That depends for example on the central bank's estimate of the reversal rate and its views about the impact of low interest rates on financial stability. Moreover, central banks may have greater resources at their disposal than markets for an assessment of these factors. Therefore, lower bound uncertainty should fall when central banks publish their estimates for the level of the lower bound. For example, the Bank of Canada (Witmer and Yang, 2016) and the Czech National Bank (Kolcunova and Havranek, 2018) have published estimates for the level of the lower bound and the associated uncertainty. The lower bound in Canada is estimated to lie between $-0.75\%$ and $-0.25\%$, while the corresponding range for the Czech Republic is $-2\%$ to $-0.4\%$.

What are the implications of a decline in lower bound uncertainty for longer-term bond yields? To answer this question, this paper studies the effects of lower bound uncertainty in a simple two-period shadow rate term structure model. The lower bound and shadow short rates are assumed to be independent random variables from the perspective of market participants. While the lower bound reflects structural factors such as the costs of storing cash, the shadow short rate is determined by monetary policy considerations. The main result is that a decrease in lower bound uncertainty – in the sense of a mean-preserving contraction of the lower bound distribution – lowers expected future short-term rates, and hence, via the expectations hypothesis of the term structure, also lowers longer-term bond yields. The intuition for this result is as follows. The short-term interest rate is the maximum of the lower bound and the shadow short rate. For any realization of the shadow short rate, this maximum is a convex function of the lower bound. Therefore it follows from Jensen's inequality that the expected value of the short-term interest rate falls when lower bound uncertainty declines.

An implication of this result for monetary policy is that communication that lowers market par-

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[1] See the discussion in Alsterlind et al. (2015), McAndrews (2015), or Witmer and Yang (2016). The extent to which such frictions are relevant depends on country-specific institutional details of the financial markets, for example, the importance of money market mutual funds.

[2] See for example "BoE's Carney says negative rates not an option for the UK: Central Banking", Reuters, August 19, 2019, available at https://www.reuters.com/article/us-britain-economy-carney/boes-carney-says-negative-rates-not-an-option-for-uk-central-banking-idUSKCN1V90R9.

ticipants' uncertainty about the lower bound can decrease longer-term yields. I believe this is a new dimension for central bank communication that has not been highlighted before. Because the lower bound varies only gradually over time (if at all), central bank communication about the lower bound should be used sparingly and cannot be considered as an instrument of monetary policy. Arguably, resolving uncertainty about the level of the lower bound, where possible, would in any case be beneficial in improving financial market efficiency. The results in this paper suggest that resolving lower bound uncertainty would also lead to a modest decline in long-term yields.

The model has a closed form solution under the assumption that shadow short rates are Gaussian, following much of the literature, and that the lower bound has a uniform distribution. With these assumptions, the effect of lower bound uncertainty on the variance of future short-term rates – and hence, in some models, on the term premium – is not clear-cut. When the shadow short rate is expected to be well above the lower bound, an increase in lower bound uncertainty makes very low values for the short-term interest rate less likely and thus lowers its variance. By contrast, when the probability that the lower bound will bind in the next period is sufficiently high, an increase in lower bound uncertainty raises the variance of the short-term interest rate. It follows that in models where the term premium depends on the variance of future short-term interest rates, the effect of a fall in lower bound uncertainty on long-term interest rates is ambiguous: while the expectations component of yields declines, the term premium may increase or decrease.

Canadian data is used to calibrate the model because Canada is one of the few countries where estimates from the central bank for the distribution of the lower bound are available. The calibration suggests that when the mean of the shadow short rate lies close to or above the mean of the lower bound, the effect of a decline in lower bound uncertainty on bond yields is dominated by the change in expected future short rates, with very little change in the term premium. Conversely, when the mean shadow rate is well below the mean lower bound, the decline in yields is mainly driven by a lower term premium. The effects are largest when the expected shadow rate is close to the lower bound, and small when it is well above the lower bound. When the range of possible lower bound values narrows by one percentage point, leaving the mean unchanged, one-year treasury bill yields decline between 3 and 5 basis points when shadow short rates are expected to be within 0.5 percentage points of the mean of the lower bound. When the lower bound range narrows by two percentage points, the corresponding decline in one-year yields is between 8 and 12 basis points. The effects on long-term yields are larger when both lower bound uncertainty and the mean of the lower bound distribution decline.

This paper is related to a number of earlier studies that look at the implications of the lower bound for the term structure of interest rates. In a theoretical model with a known and constant lower bound of zero, Ruge-Murcia (2006) finds that the effect of short-term interest rates on longer-term yields weakens when interest rates approach the lower bound. Grisse et al. (2017) extend this model to the case of a nonzero lower bound and show that an unexpected and permanent decline in the market-perceived lower bound is associated with a fall in long-term interest rates. This paper uses the same theoretical setting as these earlier papers to study the effects of lower bound uncertainty. Tillmann (2021) also studies lower bound uncertainty but focuses on its implications for robust monetary policy in a model with only short-term bonds. This paper instead studies the implications of lower bound uncertainty for longer-term bond yields. A number of empirical term structure models have been put forward where the lower bound is an estimated parameter that changes over time (Kortela (2016), Lemke and Vladu (2017), Wu and Xia (2020)). These papers do not explicitly model or discuss the effects of lower bound uncertainty.

This paper also relates to a strand of the literature that explores the macroeconomic effects of uncertainty, including, in particular, monetary policy uncertainty. Jordà and Salyer (2003) study the

implications of monetary policy uncertainty for the term structure in a limited participation model. In this setting, higher uncertainty about money growth is associated with lower interest rates. For longer maturities, this result follows because marginal utility is a convex function of money growth, so that an increase in uncertainty raises expected marginal utility. Nakata (2017) studies the interaction of the lower bound and macroeconomic uncertainty in a DSGE model. In his model, uncertainty affects macroeconomic variables because, due to the lower bound, the policy rate is a convex function while firms' marginal costs are a concave function of shocks. Therefore, higher uncertainty in the sense of a mean-preserving spread of the shock distribution raises expected interest rates and lowers expected marginal costs. Tillmann (2020) empirically studies how the response of the yield curve to monetary policy shocks depends on monetary policy uncertainty.

The remainder of this paper is structured as follows. The next section presents a simple shadow rate term structure model and studies the effects of lower bound uncertainty based on the expectations hypothesis. Section 3 extends the model to explore the effects of lower bound uncertainty on the term premium. Section 4 calibrates the model to assess the quantitative importance of lower bound uncertainty.

## 2 Lower bound uncertainty and the expectations hypothesis

### 2.1 The model

The model is a standard two-period shadow rate term structure model, based on Black (1995) and Ruge-Murcia (2006), and extends that setting to the case of a nonzero and uncertain lower bound on nominal interest rates.[3] The short-term nominal interest rate $r_t$ is constrained by the effective lower bound $\bar{r}$,

$$r_t = \max\left(r_t^*, \bar{r}\right) \tag{1}$$

where $r_t^*$ is the "shadow" short rate that prevails above the lower bound but is unobserved when $r_t^* < \bar{r}$. The shadow rate follows

$$r_{t+1}^* = \mathbb{E}_t\left(r_{t+1}^*\right) + \sigma_\varepsilon \varepsilon_{t+1} \tag{2}$$

where $\varepsilon_t$ is a zero-mean shock, with the normalization $\sigma_\varepsilon > 0$. I also assume that $\varepsilon_{t+1}$ and $\bar{r}$ (and hence $r_{t+1}^*$ and $\bar{r}$) are independent. This assumption implies that there is no feedback from the lower bound to the shadow short rate process, i.e., better information about the lower bound does not change the outlook for monetary policy. On the one hand, the lower bound is determined by the structural factors mentioned above – including the costs of storing cash and the reversal rate. These factors reflect institutional features of the financial system that change only slowly, and should be independent of the outlook for monetary policy. On the other hand, the path for the shadow short rate reflects monetary policy considerations at business cycle frequency. The desired stance of monetary policy should not depend on the structural, low-frequency determinants of the lower bound. The assumption rules out a scenario where the central bank changes its policy rate path in response to information about the lower bound.

Finally, I assume that the expectations hypothesis of the term structure holds, so that the yield of a 2-period bond $R_t$ is given by

$$R_t = \frac{1}{2}r_t + \frac{1}{2}\mathbb{E}_t(r_{t+1}) + \theta \tag{3}$$

where $\theta$ is the term premium. For now I follow Ruge-Murcia (2006) and do not explicitly model the term

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[3] Shadow rate models have become popular in term structure modeling in the last few years. See for example Kim and Singleton (2012), Bauer and Rudebusch (2016), Christensen and Rudebusch (2016), Wu and Xia (2016) and Lemke and Valdu (2017).

premium. Section 3 extends the model to study how lower bound uncertainty affects the term premium.

The new element of the model is that the lower bound $\bar{r}$ is uncertain. Beliefs about $\bar{r}$ may change over time, for example because of the publication of research that provides a more precise estimate of $\bar{r}$ than previously available. The effects on long-term yields will, in general, depend on the particular form of the probability distributions of both $\bar{r}$ and $r_{t+1}^*$. Section 2.2 discusses the effects of a reduction in lower bound uncertainty without specifying the probability distributions of $\bar{r}$ and $r_{t+1}^*$. Section 2.3 then introduces particular assumptions for these distributions that allow a closed-form solution of the model.

## 2.2 A result for general distributions of shadow short rates and lower bound

Suppose that from the perspective of market participants, the lower bound is initially – conditional on an information set $I_h$ – a random variable $\bar{r}_h$. After lower bound uncertainty is reduced, for example due to central bank communication, the information set is $I_\ell$, with $I_\ell \supset I_h$, and conditional on $I_\ell$ the lower bound is now a random variable $\bar{r}_\ell$. These random variables are related by

$$\bar{r}_h = \bar{r}_\ell + \eta \tag{4}$$

where

$$\mathbb{E}\left(\eta | \bar{r}_\ell, r_{t+1}^*\right) = 0 \tag{5}$$

The random variables $\bar{r}_\ell, \eta$ and $r_{t+1}^*$ are independent, with $\mathbb{E}_t\left(\bar{r}_h\right) = \mathbb{E}_t\left(\bar{r}_\ell\right) = \mathbb{E}_t\left(\bar{r}\right)$. The transformation of the distribution of $\bar{r}_\ell$ to the distribution of $\bar{r}_h$ in (4) and (5) is a mean-preserving spread (Rothschild and Stiglitz (1970)).[4] The reduction in uncertainty from $I_h$ to $I_\ell$ is a mean-preserving contraction. Note that the variance of $\bar{r}$ is lower under $I_\ell$ than under $I_h$, since $\mathbb{V}\left(\bar{r}|I_h\right) = \mathbb{V}\left(\bar{r}_\ell\right) + \mathbb{V}\left(\eta\right) > \mathbb{V}\left(\bar{r}_\ell\right) = \mathbb{V}\left(\bar{r}|I_\ell\right)$.[5] I make no further assumptions on the distribution of $\bar{r}$. In particular, this distribution does not need to be symmetric, and $\mathbb{E}_t\left(\bar{r}\right)$ could be above or below $\mathbb{E}_t\left(r_{t+1}^*\right)$.

**Proposition 1** *Under a mean-preserving contraction of the distribution of the lower bound, expected future short-term interest rates do not increase.*

**Proof.** Conditional on some realization of $r_{t+1}^*$ and $\bar{r}_\ell$, we have

$$\mathbb{E}\left[\max\left(r_{t+1}^*, \bar{r}_\ell + \eta\right) | r_{t+1}^*, \bar{r}_\ell\right] \geq \max\left(r_{t+1}^*, \bar{r}_\ell + \mathbb{E}\left(\eta | r_{t+1}^*, \bar{r}_\ell\right)\right)$$
$$= \max\left(r_{t+1}^*, \bar{r}_\ell\right)$$

The weak inequality follows from Jensen's inequality and the fact that $\max\left(r_{t+1}^*, \bar{r}_\ell + \eta\right)$ is a convex function of $\eta$. The second line uses assumption (5). Using the law of iterated expectations we have

$$\mathbb{E}\left(r_{t+1}|I_h\right) = \mathbb{E}\left\{\mathbb{E}\left[\max\left(r_{t+1}^*, \bar{r}_\ell + \eta\right) | r_{t+1}^*, \bar{r}_\ell\right]\right\}$$
$$\geq \mathbb{E}\left[\max\left(r_{t+1}^*, \bar{r}_\ell\right)\right]$$
$$= \mathbb{E}\left(r_{t+1}|I_\ell\right)$$

which completes the proof. ∎

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[4] Mean-preserving spreads have been widely used to study the macroeconomic effects of uncertainty, see e.g. Jordà and Salyer (2003), Christiano, Motto and Rostagno (2014), Gilchrist, Sim and Zakrajsek (2014), Nakata (2017) and Arellano, Bai and Kehoe (2019).

[5] A mean-preserving spread implies a higher variance, but the reverse is not true. In particular, an increase in the variance of the lower bound does not necessarily imply that expected future short-term rates increase, as in proposition 1.

This result implies that a reduction in uncertainty about $\bar{r}$ – in the sense of a mean-preserving contraction of the distribution of $\bar{r}$ – lowers long-term bond yields via the expectations component. A policy implication is that credible central bank communication about the effective lower bound can influence longer-term bond yields. Because the short-term interest rate, as the maximum of $r^*_{t+1}$ and $\bar{r}$, is a convex function of the lower bound, a mean-preserving contraction of the distribution of $\bar{r}$ lowers $\mathbb{E}\left(\max\left(r^*_{t+1}, \bar{r}\right) | r^*_{t+1}\right)$. Since this is true for any given $r^*_{t+1}$, it is also true in expectation across all $r^*_{t+1}$, and hence the expected short rate declines when lower bound uncertainty falls. A mean-preserving contraction can be thought of as having two effects. On the one hand, it makes very low values of $\bar{r}$ less likely. This effect raises $\mathbb{E}\left(r_{t+1}\right)$. On the other hand, it also makes very high values of $\bar{r}$ less likely. This effect lowers $\mathbb{E}\left(r_{t+1}\right)$. Due to the convexity of $\max\left(r^*_{t+1}, \bar{r}_\ell\right)$, the second effect dominates, so that expected short rates decline when lower bound uncertainty falls.

## 2.3   Model solution and implications

This section introduces assumptions about the distributions of shadow short rates and the lower bound that allow a closed form solution of the model. This closed-form solution will be used below to derive some additional results on the effects of lower bound uncertainty, to study the response of the term premium to lower bound uncertainty in section 3 and to explore the quantitative relevance of lower bound uncertainty in section 4.

I follow Ruge-Murcia (2006) and much of the literature and assume that future shadow short rates are normally distributed conditional on the information available to market participants, i.e., the shadow short rate shock in (2) follows $\varepsilon_t \sim N(0,1)$. I also assume that $\bar{r}$ has a uniform distribution on the interval $\bar{\mu} \pm \bar{\sigma}$. The assumption that $\bar{r}$ is uniformly distributed has two advantages. First, the model admits a closed-form solution. Second, the limited support of the uniform distribution captures the fact that the range of possible $\bar{r}$ values is bounded above by zero, by the lowest historically observed short-term rate, or by central bank communication about $\bar{r}$ and is bounded from below by the fact that holding cash will eventually dominate holding negative yielding bonds once interest rates become sufficiently negative. This assumption also implies that the distribution of $\bar{r}$ is symmetric. This may not necessarily be a realistic assumption. However, it is convenient because it allows to separate the effect of a change in uncertainty about $\bar{r}$ from that of a change in the mean of $\bar{r}$.

To solve the model, we must compute market expectations of future short-term interest rates,

$$\mathbb{E}_t(r_{t+1}) = \mathbb{E}_t\left[\max\left(r^*_{t+1}, \bar{r}\right)\right]$$

We can write

$$\begin{aligned}\max\left(r^*_{t+1}, \bar{r}\right) &= \max\left[\mathbb{E}_t\left(r^*_{t+1}\right) + \sigma_\varepsilon \varepsilon_{t+1}, \bar{r}\right] \\ &= \mathbb{E}_t\left(r^*_{t+1}\right) + \sigma_\varepsilon \max\left(\varepsilon_{t+1}, \bar{\rho}\right)\end{aligned} \qquad (6)$$

where

$$\bar{\rho} \equiv \frac{\bar{r} - \mathbb{E}_t\left(r^*_{t+1}\right)}{\sigma_\varepsilon}$$

It follows that

$$\mathbb{E}_t\left(r_{t+1}\right) = \mathbb{E}_t\left(r^*_{t+1}\right) + \sigma_\varepsilon \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right]$$

Given the assumption that $\bar{r}$ has a uniform distribution over the interval $\bar{\mu} \pm \bar{\sigma}$, the "adjusted lower

bound" $\bar{\rho}$ has a uniform distribution on the interval $\mu \pm \sigma$, where

$$\mu \equiv \frac{\bar{\mu} - \mathbb{E}_t\left(r_{t+1}^*\right)}{\sigma_\varepsilon}$$

$$\sigma \equiv \frac{\bar{\sigma}}{\sigma_\varepsilon}$$

**Proposition 2** *The mean and variance of the shock to the short-term interest rate in $t + 1$, conditional on information available in $t$, are*

$$\mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] = \frac{1}{4\sigma}\left[\left(1 + b^2\right)\Phi\left(b\right) - \left(1 + a^2\right)\Phi\left(a\right) + b\phi\left(b\right) - a\phi\left(a\right)\right] \tag{7}$$

$$\mathbb{V}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] = 1 + \frac{1}{6\sigma}\left\{ \begin{array}{c} \left(b^2 - 3\right)b\Phi\left(b\right) - \left(a^2 - 3\right)a\Phi\left(a\right) \\ + \left(b^2 - 4\right)\phi\left(b\right) - \left(a^2 - 4\right)\phi\left(a\right) \end{array} \right\} - \left\{\mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right]\right\}^2 \tag{8}$$

*where $b \equiv \mu + \sigma$, $a \equiv \mu - \sigma$ and $\phi\left(\cdot\right)$ and $\Phi\left(\cdot\right)$ denote the standard normal pdf and cdf.*

**Proof.** See appendix. ∎

Due to the presence of the lower bound, the mean of future short rate shocks in (7) is positive, $\mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] > 0$. In the absence of the lower bound, or when the lower bound becomes non-binding (in the limit as $\bar{\mu} \to -\infty$, and hence $\mu \to -\infty$), we have $\mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] \to \mathbb{E}_t\left(\varepsilon_{t+1}\right) = 0$. In the limit where $\bar{\sigma} \to 0$ and hence $\sigma \to 0$, the lower bound is known, and (7) simplifies to the formula given in Grisse et al (2017), or in Ruge-Murcia (2006) when $\bar{\mu} = 0$. Regarding the variance, when $\bar{\mu} \to -\infty$ the lower bound becomes irrelevant, and hence the variance of the short rate tends towards the variance of the shadow rate, $\mathbb{V}_t(r_{t+1}) \to \mathbb{V}_t\left(r_{t+1}^*\right) = \sigma_\varepsilon^2$. Conversely, when $\bar{\mu} \to \infty$, the lower bound is expected to be binding in $t + 1$ with certainty, and consequently the variance of the short rate tends to the variance of the lower bound, $\mathbb{V}_t(r_{t+1}) \to \mathbb{V}_t\left(\bar{r}\right) = \bar{\sigma}^2/3$.

Now we are ready to explore how changes in market-perceived uncertainty about the lower bound affect the level of long-term interest rates and the link between short- and long-term interest rates. Note that a decline in $\bar{\sigma}$ corresponds to a mean-preserving contraction of the distribution of $\bar{r}$, as analyzed in proposition 1.

**Proposition 3** *Under a mean-preserving contraction of the lower bound distribution, expected future short-term interest rates decline. The magnitude of this effect tends to zero as $\mu \to \pm\infty$, is increasing in $\mu$ when $\mu < 0$, reaches a maximum when $\mu = 0$, and is decreasing in $\mu$ when $\mu > 0$.*

**Proof.** See appendix. ∎

This result is consistent with proposition 1 and additionally shows how the effects of lower bound uncertainty vary with $\mu$, i.e., the degree to which the lower bound is expected to be binding in the next period. To build intuition, consider the case where the lower bound is expected to be nonbinding in the next period, in the sense that $\mathbb{E}_t\left(r_{t+1}^*\right) > \bar{\mu}$ or equivalently $\mu < 0$. An increase in uncertainty around $\bar{\mu}$ then has two effects. First, it implies that lower values are possible for $\bar{r}$ than previously thought, which on its own lowers $\mathbb{E}_t\left(r_{t+1}\right)$. Second, it equally means that higher values are possible for $\bar{r}$ than previously thought, which on its own raises $\mathbb{E}_t\left(r_{t+1}\right)$. Overall, $\mathbb{E}_t\left(r_{t+1}\right)$ increases because the second effect dominates. The reason is that the possibility of a very low $\bar{r}$ does not raise interest rates very

much, because it is relevant only in the unlikely case of a very low realization of the shadow short rate, in the left tail of the normal distribution. In the case where the lower bound is expected to be binding, in the sense that $\mathbb{E}_t\left(r_{t+1}^*\right) < \bar{\mu}$ and hence $\mu > 0$, the intuition is similar. Again, an increase in lower bound uncertainty has two effects: lower values are possible for $\bar{r}$ than previously thought, lowering $\mathbb{E}_t\left(r_{t+1}\right)$, but higher values for $\bar{r}$ are possible as well, raising $\mathbb{E}_t\left(r_{t+1}\right)$. Again the second effect dominates. If $\bar{r}$ is low, $r_{t+1}$ will only be lower than previously thought possible if a low value for $r_{t+1}^*$ is realized. By contrast, if $\bar{r}$ turns out to be high, $r_{t+1}$ will increase at least one-for-one with $\bar{r}$.

Proposition 3 also establishes that the effect of lower bound uncertainty is strongest when $\mu = 0$, so that the distributions of lower bound and shadow short rate have the same mean. When $\mu$ is very large, such that the lower bound is binding with a high probability, $\mathbb{E}_t\left(r_{t+1}\right) \simeq \mathbb{E}_t\left(\bar{r}\right)$ and hence lower bound uncertainty does not affect $\mathbb{E}_t\left(r_{t+1}\right)$. The same is true when $\mu$ is very small, such that $\mathbb{E}_t\left(r_{t+1}\right) \simeq \mathbb{E}_t\left(r_{t+1}^*\right)$. It is therefore intuitive that the effect of lower bound uncertainty on $\mathbb{E}_t\left(r_{t+1}\right)$ is largest when $\mu = 0$.

Ruge-Murcia (2006) showed that the response of longer-term yields to changes in the short-term rate weakens as interest rates approach the lower bound. The next result shows how this effect depends on lower bound uncertainty.

**Proposition 4** *Suppose that $\frac{\partial}{\partial r_t}\mathbb{E}_t\left(r_{t+1}^*\right) > 0$. Then, a mean-preserving contraction of the lower bound distribution strengthens the response of expected future short-term interest rates to changes in the current short rate when $\mu < 0$, weakens the response when $\mu > 0$, and has no effect when $\mu = 0$.*

**Proof.** See appendix. ■

From (3), this result implies that the effect of short-term interest rates on longer-term bond yields varies with lower bound uncertainty. When $\mu < 0$, so that the expected shadow short rate is above the expected lower bound, a decline in lower bound uncertainty strengthens the response of $\mathbb{E}_t\left(r_{t+1}^*\right)$ to short rates. The reason is that it becomes less likely that the lower bound is binding in the next period, and hence more likely that $r_{t+1}$ will increase with $r_{t+1}^*$. Conversely, when $\mu > 0$ a decline in lower bound uncertainty raises the likelihood that the lower bound is binding, and hence makes it less likely that $r_{t+1}$ will increase with $r_{t+1}^*$.

## 3   Lower bound uncertainty and the term premium

The framework set out above is based on Ruge-Murcia (2006) and Grisse et al. (2017), and like those earlier papers, it did not consider the effect of the lower bound on the term premium. Empirical studies such as Hanson and Stein (2015) and Crump, Eusepi and Mönch (2018) have shown, however, that interest rate movements are often dominated by changes in the term premium, and that the term premium responds to monetary policy announcements. Therefore, it makes sense to extend the analysis by considering a simple model for the term premium.

To do this, I follow Hanson and Stein (2015) and Barillas and Nimark (2017), among others, and use a simple mean-variance framework to obtain an expression for the term premium. In period $t$, the representative investor chooses a portfolio of one-period bonds (with log return $r_t$) and two-period bonds (with log yield $R_t$) to maximize

$$\mathbb{E}_t\left(w_{t+1}\right) - \frac{1}{2}\gamma\mathbb{V}_t\left(w_{t+1}\right) \tag{9}$$

where $\gamma > 0$ measures the investor's risk aversion and $w_t$ is wealth. If the investor purchases $b_t$ two-period bonds and finances this position by rolling over short-term debt, his wealth in $t+1$ is given by

$$w_{t+1} = b_t \left( 2R_t - r_t - r_{t+1} \right)$$

Substituting this into (9), the first-order condition gives the optimal holdings of long-term bonds as

$$b_t^* \left( R_t \right) = \frac{2R_t - r_t - \mathbb{E}_t \left( r_{t+1} \right)}{\gamma \mathbb{V}_t \left( r_{t+1} \right)}$$

In equilibrium, the demand for bonds $b_t^* \left( R_t \right)$ must equal supply $Q$. Rearranging this market clearing condition, we get

$$R_t = \frac{1}{2} r_t + \frac{1}{2} \mathbb{E}_t \left( r_{t+1} \right) + \frac{1}{2} \gamma \mathbb{V}_t \left( r_{t+1} \right) Q \tag{10}$$

The model therefore generates the same equation for the (log) bond yield as (3), with a term premium given by $\theta = \frac{1}{2} \gamma \mathbb{V}_t \left( r_{t+1} \right) Q$. The term premium compensates investors for the risk of holding two-period bonds. This risk premium is higher if investors are more risk averse (higher $\gamma$), bonds are more risky (higher $\mathbb{V}_t \left( r_{t+1} \right)$) or there are more bonds outstanding (higher $Q$). From (6), the variance of next period's short-term interest rate is

$$\mathbb{V}_t \left( r_{t+1} \right) = \sigma_\varepsilon^2 \mathbb{V}_t \left[ \max \left( \varepsilon_{t+1}, \bar\rho \right) \right] \tag{11}$$

where $\mathbb{V}_t \left[ \max \left( \varepsilon_{t+1}, \bar\rho \right) \right]$ was computed in (8). Then

$$\frac{\partial \mathbb{V}_t \left( r_{t+1} \right)}{\partial \bar\sigma} = \sigma_\varepsilon^2 \frac{\partial \mathbb{V}_t \left[ \max \left( \varepsilon_{t+1}, \bar\rho \right) \right]}{\partial \sigma} \frac{\partial \sigma}{\partial \bar\sigma} = \sigma_\varepsilon \frac{\partial \mathbb{V}_t \left[ \max \left( \varepsilon_{t+1}, \bar\rho \right) \right]}{\partial \sigma}$$

While a closed form expression for this derivative exists, it is difficult to show analytically under which conditions it is positive.[6] Therefore, I explore numerically how this effect varies with $\mu$ and $\sigma$. I report $1/\sigma_\varepsilon$ times the derivative of interest in order to analyze how the result depends on the relative importance of uncertainty about $\bar r$ and $r_{t+1}^*$, without making an assumption about $\sigma_\varepsilon$.

Figure 1 reports the results. Recall that positive values for $\mu$ indicate that the lower bound is expected to be binding in the next period. Furthermore, since $\sigma \equiv \bar\sigma / \sigma_\varepsilon$ and $\mathbb{V}_t \left( \bar r \right) = \bar\sigma^2 / 3$, values of $\sigma$ below $\sqrt{3} \approx 1.7$ imply that $\mathbb{V}_t \left( \bar r \right) < \mathbb{V}_t \left( r_{t+1}^* \right)$. Panel (a) shows that $\partial \mathbb{V}_t \left( r_{t+1} \right) / \partial \bar\sigma$ is negative when $\mu$ and $\sigma$ are sufficiently small. In this case, with the lower bound expected to be nonbinding, an increase in $\bar\sigma$ makes very low values of the short rate less likely and thus lowers $\mathbb{V}_t \left( r_{t+1} \right)$. When $\mu$ is positive, $\mathbb{V}_t \left( r_{t+1} \right)$ is always increasing in $\bar\sigma$. This can also be seen in panel (b), which shows how $\partial \mathbb{V}_t \left( r_{t+1} \right) / \partial \bar\sigma$ varies with $\mu$. When $\mu$ is sufficiently high, such that the lower bound is expected to be binding with a high probability and the distributions of $\bar r$ and $r_{t+1}^*$ have a large amount of overlap, an increase in $\bar\sigma$ raises $\mathbb{V}_t \left( r_{t+1} \right)$ by making very low values of $r_{t+1}$ possible, and very high values of $r_{t+1}$ more likely (if $\bar\mu$ is high enough). When $\mu$ is well above zero, $\partial \mathbb{V}_t \left( r_{t+1} \right) / \partial \bar\sigma$ approaches some constant positive value: when there is a high probability that short rates next period are equal to $\bar r$, then $\mathbb{V}_t \left( r_{t+1} \right) \approx \mathbb{V}_t \left( \bar r \right)$ and therefore $\partial \mathbb{V}_t \left( r_{t+1} \right) / \partial \bar\sigma \approx \partial \mathbb{V}_t \left( \bar r \right) / \partial \bar\sigma$, which does not depend on $\mu$. Conversely, in the limit where $\mu \to -\infty$ the lower bound becomes irrelevant and $\partial \mathbb{V}_t \left( r_{t+1} \right) / \partial \bar\sigma \approx \partial \mathbb{V}_t \left( r_{t+1}^* \right) / \partial \bar\sigma = 0$. Overall, the effect of lower bound uncertainty on the variance of the short-term interest rate, and hence the term premium, is ambiguous.

---

[6] Note that the logic of proposition 1 does not carry over to the variance of future short-term interest rates. The reason is that while the $\mathbb{V} \left( \cdot \right)$ and $\max \left( \cdot, \cdot \right)$ are both convex functions, $\mathbb{V} \left[ \max \left( \cdot, \cdot \right) \right]$ as a convex function of a convex function is not generally convex.
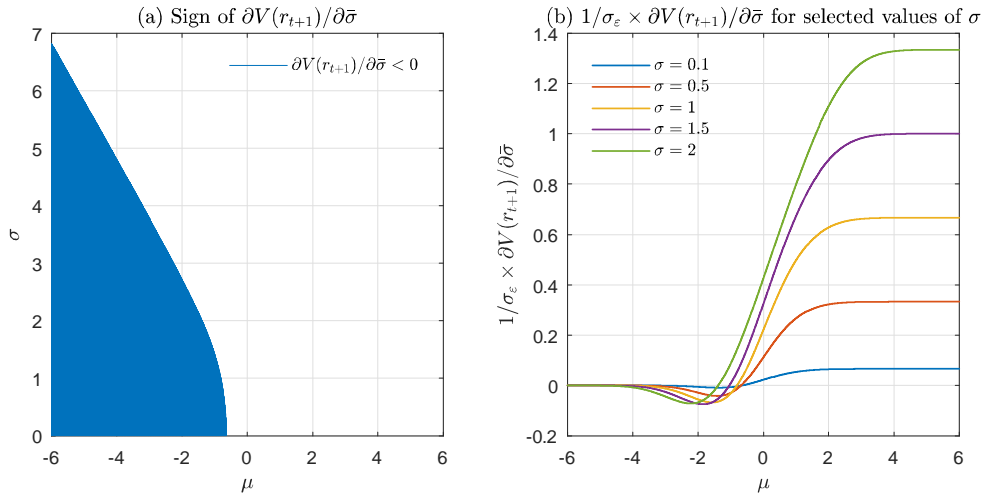
Figure 1: Effect of lower bound uncertainty on $\frac{1}{\sigma_\varepsilon}\mathbb{V}_t\left(r_{t+1}\right)$. An increase in $\mu$ corresponds to a higher market-perceived probability that the lower bound is binding. A decline in $\bar{\sigma}$ corresponds to a mean-preserving contraction in the distribution of $\bar{r}$.

# 4  Is lower bound uncertainty quantitatively relevant?

Because it is challenging to empirically disentangle the effects of uncertainty about the lower bound from effects of uncertainty about shadow short rates, this paper has focused on qualitative results using a tractable theoretical framework. Nevertheless, this section tries to give some idea about the magnitude and economic relevance of these effects.

Witmer and Yang (2016) estimate that the lower bound in Canada is between $-0.25\%$ and $-0.75\%$, with a median of $-0.5\%$. I take these estimates as a starting point and assume that within this range, $\bar{r}$ is uniformly distributed, as in the model, with $\bar{\mu} = -0.5$ and $\bar{\sigma} = 0.25$. To calculate the standard deviation of shadow short rate shocks $\sigma_\varepsilon$, I use Consensus expectations about the Canadian 3-month treasury bill, one and four quarters ahead, and calculate the standard deviation of the difference between those expectations and the ex-post realized interest rates 3-months and 12-months after the survey. To isolate uncertainty about future (shadow) short rates, rather than the lower bound, I focus on the period from January 2000 to December 2007. During this period, the minimum value of the expected 3-month rate was 1.8% one quarter ahead, and 2.5% four quarters ahead. Interest rates were thus expected to be well above zero in this period, so that lower bound uncertainty was of little practical relevance. This gives an estimate of $\sigma_\varepsilon \approx 0.50$ one quarter ahead and $\sigma_\varepsilon \approx 1.15$ four quarters ahead. As expected, the average absolute forecast error increases with the forecast horizon.[7]

For the term premium, I assume that (10) holds and that the Consensus expectations are representative of the beliefs held by traders in the Canadian government bond market. I then calculate the term premium as the difference between 6-month or 1-year bond yields and the expectations component implied by Consensus expectations.[8] This gives an average term premium over the period from January 2000 to December 2007 of $\theta \approx 0.03$ for 6-month yields and $\theta \approx 0.06$ for 1-year yields. I then assume, as in (10), that $\theta = \frac{1}{2}\gamma\mathbb{V}_t\left(r_{t+1}\right)Q$. Under a change of the information set on which lower bound beliefs are based from $I_0$ to $I_1$, the variance of the short-term interest rate changes from $\mathbb{V}_t\left(r_{t+1}|I_0\right)$ to $\mathbb{V}_t\left(r_{t+1}|I_1\right)$.

---

[7] I interpolate these estimates to obtain values of $\sigma_\varepsilon$ for the two- and three-quarter ahead horizons, which are needed to calculate the effects on the expectations component of 1-year bond yields.

[8] This approach of calculating the term premium based on survey expectations follows Crump, Eusepi and Mönch (2018).
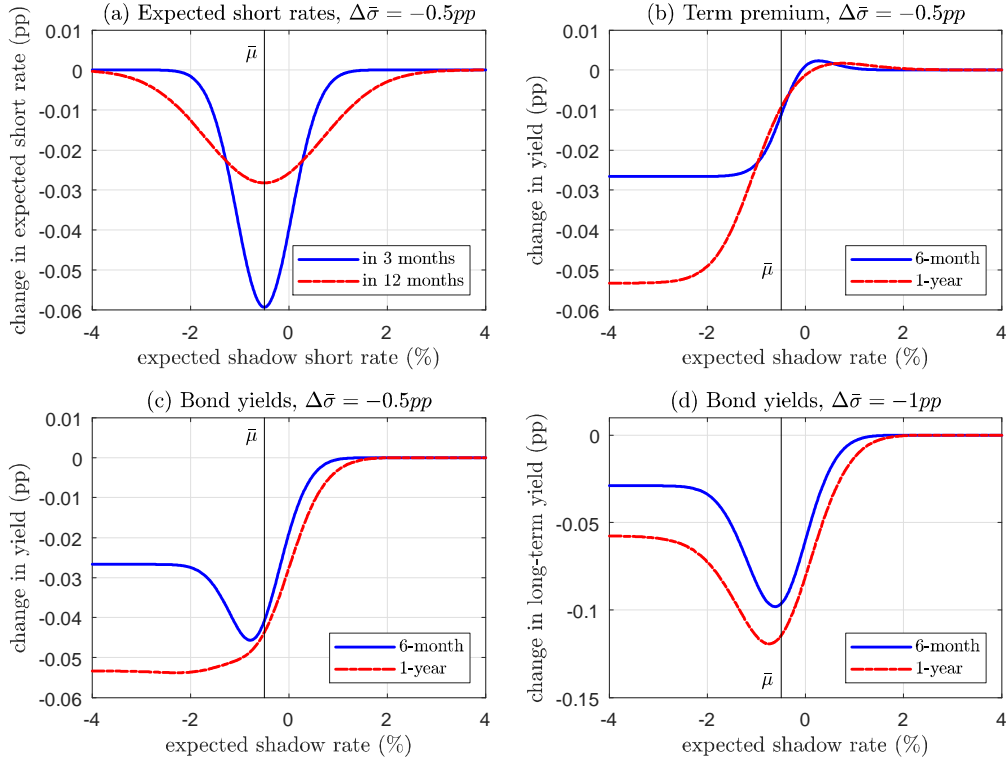
Figure 2: Calibration of the effect of a change in $\bar{\sigma}$ on expected future short rates, the term premium and bond yields.

I calculate the associated change in the term premium as

$$
\begin{aligned}
\Delta\theta &= \frac{1}{2}\gamma Q\left[\mathbb{V}_t\left(r_{t+1}|I_1\right) - \mathbb{V}_t\left(r_{t+1}|I_0\right)\right] \\
&= \theta\left[\frac{\mathbb{V}_t\left(r_{t+1}|I_1\right)}{\mathbb{V}_t\left(r_{t+1}|I_0\right)} - 1\right]
\end{aligned}
$$

where $\theta = \frac{1}{2}\gamma Q\mathbb{V}_t\left(r_{t+1}|I_0\right)$ is the average term premium in the data.[9]

I use these estimates to explore how a one percentage point decline in the range of possible values for $\bar{r}$, from $[-1.25, 0.25]$ to $[-0.75, -0.25]$ with an unchanged mean, affects interest rates. To do this, I simply plug in the estimates for $\sigma_\varepsilon$, $\bar{\sigma}$ and $\bar{\mu}$ into the closed-form expression for expected future short rates in equation (7), and compute how the effects depend on the level of expected future short rates, $\mathbb{E}_t\left(r_{t+j}^*\right)$. Panel (a) in Figure 2 shows how the resulting change in expected future short-term rates, $\Delta\mathbb{E}_t\left(r_{t+j}\right)$, depends on the level of expected future shadow short rates $\mathbb{E}_t\left(r_{t+j}^*\right)$. For expected short rates 12 months ahead, I assume a constant expected shadow short rate over the entire horizon (3, 6 and 9 months ahead). Panel (b) shows the corresponding change in the term premium, and panel (c) reports the resulting change in bond yields, as the sum of the change in the expectations component and the term premium. The change in the expectations component is larger for 1-year yields than for 6-month yields, because current (unchanged) 3-month rates receive a smaller weight. It is largest when the mean shadow rate is close to the mean lower bound $\bar{\mu}$, in line with proposition 3. The change in

---

[9]For calculating the change in the term premium of 1-year yields, I simply use the change in the variance of the 3-month rate 9 months ahead to avoid making assumptions about the stochastic process of $r_t^*$.

the term premium is largest when the mean shadow rate is well below $\bar{\mu}$, so that $\mathbb{V}_t(r_{t+1}) \approx \mathbb{V}_t(\bar{r})$ and lower bound uncertainty has a large effect on $\mathbb{V}_t(r_{t+1})$. Overall, the effects on bond yields are small and nonnegligible only when expected shadow short rates are close to or below the mean lower bound. For example, when the shadow short rate is expected to be 0.25 percent 3, 6 and 9 months ahead – the initial upper end of the range of possible lower bound values (at $\bar{\mu} + \bar{\sigma}$) – 1-year yields decline by just 2 basis points as lower bound uncertainty falls. In this calibration, the decline in yields is mainly driven by the expectations component in the more relevant scenario where shadow rates are close to or above the mean lower bound $\bar{\mu}$, and mainly by the term premium when the mean shadow rate is well below $\bar{\mu}$.

A narrowing of the range of possible $\bar{r}$ values by one percentage point, however, is a relatively modest change. Panel (d) therefore considers a one percentage point decline in $\bar{\sigma}$, so that the range of possible $\bar{r}$ values narrows by two percentage points.[10] In this case 1-year bond yields decline by up to 12 basis points. Finally, note that central bank communication likely changes not only lower bound uncertainty but may shift the whole distribution of $\bar{r}$ values. This can generate much larger effects on yields.

# 5    Conclusion

The lower bound on nominal interest rates is a standard element in many models but is typically treated as a known parameter. In practice, however, the level of the lower bound is uncertain. This paper explores the implications of lower bound uncertainty in a simple shadow rate term structure model. I show that a decline in lower bound uncertainty is associated with lower expected short-term interest rates, and therefore, through the expectations hypothesis of the term structure, is also associated with lower long-term yields. The effect of lower bound uncertainty on the variance of short-term interest rates and hence, in some models, on the term premium is ambiguous. The findings in this paper suggest that central bank communication about lower bound uncertainty can influence the term structure of interest rates.

Overall, the contribution of this paper is to study the implications of lower bound uncertainty in the context of a simple and a tractable model. Rather than aiming for a model that can be taken to the data, the focus is on the qualitative effects of lower bound uncertainty. This is a useful first step to better understand the implications of lower bound uncertainty for financial markets and monetary policy. The next step of estimating the effects of lower bound uncertainty from the data is challenging. It is difficult to extract market views about the lower bound from interest rates or options prices because these data reflect beliefs about the lower bound as well as about the monetary policy outlook. At the same time, central bank communication regarding the lower bound will usually affect not only lower bound uncertainty, but also the mean of the perceived lower bound distribution, as well as the outlook for monetary policy. Moreover, without a measure of market-perceived lower bound uncertainty, it is not clear whether central bank announcements increase or decrease market-perceived lower bound uncertainty – and hence, it is not clear what interest rate response we should expect following such announcements.

A possible empirical approach would be to conduct surveys among financial analysts regarding their assessment of the lower bound, and to compute measures of lower bound uncertainty from the survey responses. The mechanism presented in the paper could then be tested by regressing bond yields on uncertainty measures, over time, in an event study of central bank communication regarding the lower bound, or across countries. The regressions would need to control for the market-perceived mean of the lower bound (computed from survey responses) and for the outlook for monetary policy.

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[10] With an unchanged mean of the lower bound of $\bar{\mu} = -0.5$, this implies an initial top end of the range of $+0.75$, which is clearly too high. However, this does not affect the conclusions regarding the distance of expected shadow short rates from the lower bound.

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# Appendix

## A    Proofs

The following properties of the standard normal pdf and cdf will be useful below.

**Lemma 5**

$$
\begin{array}{ll}
(i) & \lim_{x\to\pm\infty} x\phi(x) = 0 \\
(ii) & \lim_{x\to\pm\infty} x^2\phi(x) = 0 \\
(iii) & \lim_{x\to-\infty} x\Phi(x) = 0 \\
(iv) & \lim_{x\to-\infty} x^2\Phi(x) = 0 \\
(v) & \lim_{x\to-\infty} x^3\Phi(x) = 0
\end{array}
$$

**Proof.** (i) Using the rule of L'Hôpital, we have

$$
\begin{aligned}
\lim_{x\to\pm\infty} x\phi(x) &= \frac{1}{\sqrt{2\pi}} \lim_{x\to\pm\infty} \frac{x}{e^{x^2/2}} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{x\to\pm\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{x^2/2}} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{x\to\pm\infty} \frac{1}{xe^{x^2/2}} \\
&= 0
\end{aligned}
$$

as required.

(ii) Again using the rule of L'Hôpital, we have

$$
\begin{aligned}
\lim_{x\to\pm\infty} x^2\phi(x) &= \frac{1}{\sqrt{2\pi}} \lim_{x\to\pm\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^{x^2/2}} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{x\to\pm\infty} \frac{2x}{xe^{x^2/2}} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{x\to\pm\infty} \frac{2}{e^{x^2/2}} \\
&= 0
\end{aligned}
$$

as required.

(iii) Let $u \sim N(0,1)$. For any $x \leq 0$, we have

$$
\int_{-\infty}^{x} u\phi(u)\,du \leq x\int_{-\infty}^{x} \phi(u)\,du = x\Phi(x) \leq 0
$$

The expression on the left of the inequality sign equals

$$
\int_{-\infty}^{x} u\phi(u)\,dx = \int_{-\infty}^{\infty} u\phi(u)\,du - \int_{x}^{\infty} u\phi(u)\,du = -\int_{x}^{\infty} u\phi(u)\,du
$$

Substituting this result into the previous inequality, we get

$$
-\int_{x}^{\infty} u\phi(u)\,du \leq x\Phi(x) \leq 0
$$

The integral on the left converges to the mean of $u$, and hence to zero, as $x \to -\infty$. Therefore, we have

$$0 \leq \lim_{x \to -\infty} x\Phi(x) \leq 0$$

which implies the result.

(iv) Let $u \sim N(0,1)$. For any $x \leq 0$, we have

$$\int_{-\infty}^{x} u^2 \phi(u)\, du \geq x^2 \int_{-\infty}^{x} \phi(u)\, dx = x^2 \Phi(x) \geq 0$$

The expression on the left of the inequality sign equals

$$\int_{-\infty}^{x} u^2 \phi(u)\, du = \int_{-\infty}^{\infty} u^2 \phi(u)\, du - \int_{x}^{\infty} u^2 \phi(u)\, du = 1 - \int_{x}^{\infty} u^2 \phi(u)\, du$$

Therefore, we have

$$1 - \int_{x}^{\infty} u^2 \phi(u)\, du \geq x^2 \Phi(x) \geq 0$$

The integral on the left converges to the variance of $u$, and hence to one as $x \to -\infty$. Therefore we have

$$0 \geq \lim_{x \to -\infty} x^2 \Phi(x) \geq 0$$

which implies the result.

(v) Let $u \sim N(0,1)$. For any $x \leq 0$, we have

$$\int_{-\infty}^{x} u^3 \phi(u)\, du \leq x^3 \int_{-\infty}^{x} \phi(u)\, du = x^3 \Phi(x) \leq 0$$

The expression on the left of the inequality sign equals

$$\int_{-\infty}^{x} u^3 \phi(u)\, du = \int_{-\infty}^{\infty} u^3 \phi(u)\, du - \int_{x}^{\infty} u^3 \phi(u)\, du = - \int_{x}^{\infty} u^3 \phi(u)\, du$$

Therefore, we have

$$- \int_{x}^{\infty} u^3 \phi(u)\, du \leq x^3 \Phi(x) \leq 0$$

The integral on the left converges to the third moment of $u$, and hence to zero as $x \to -\infty$. Therefore

$$0 \leq \lim_{x \to -\infty} x^3 \Phi(x) \leq 0$$

which implies the result. ∎

**Proof of proposition 2.** The pdf $f_{\bar{\rho}}$ and cdf $F_{\bar{\rho}}$ of $\bar{\rho}$ are

$$f_{\bar{\rho}}(w) = \begin{cases} \frac{1}{2\sigma} & \text{if } \mu - \sigma < w < \mu + \sigma \\ 0 & \text{otherwise} \end{cases}$$

$$F_{\bar{\rho}}(w) = \begin{cases} 0 & \text{if } w \leq \mu - \sigma \\ \frac{w - (\mu - \sigma)}{2\sigma} & \text{if } \mu - \sigma < w < \mu + \sigma \\ 1 & \text{if } w \geq \mu + \sigma \end{cases}$$

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Let $f$ and $F$ denote the pdf and cdf of $\max(\varepsilon_{t+1}, \bar{\rho})$. Since $\varepsilon_{t+1}$ and $\bar{\rho}$ are independent, we have

$$
\begin{aligned}
F(w) &= \mathbb{P}[\max(\varepsilon_{t+1}, \bar{\rho}) \leq w] \\
&= \mathbb{P}(\varepsilon_{t+1} \leq w)\mathbb{P}(\bar{\rho} \leq w) \\
&= \Phi(w) F_{\bar{\rho}}(w)
\end{aligned}
$$

Therefore, the pdf of $\max(\varepsilon_{t+1}, \bar{\rho})$ is

$$
\begin{aligned}
f(w) &= \frac{dF(w)}{dw} \\
&= \phi(w) F_{\bar{\rho}}(w) + \Phi(w) f_{\bar{\rho}}(w) \\
&= \begin{cases} 0 & \text{if } w \leq \mu - \sigma \\ \phi(w)\frac{w-(\mu-\sigma)}{2\sigma} + \Phi(w)\frac{1}{2\sigma} & \text{otherwise} \\ \phi(w) & \text{if } w \geq \mu + \sigma \end{cases}
\end{aligned}
$$

For the mean, equation (7) in the paper, we have

$$
\mathbb{E}_t[\max(\varepsilon_{t+1}, \bar{\rho})] = \frac{1}{2\sigma}\int_{\mu-\sigma}^{\mu+\sigma}\left[w^2\phi(w) + w\Phi(w) - (\mu-\sigma)w\phi(w)\right]dw + \int_{\mu+\sigma}^{\infty} w\phi(w)\,dw
$$

Using the fact that

$$
\begin{aligned}
\frac{d}{dw}\frac{1}{2}\left[(1+w^2)\Phi(w) - w\phi(w)\right] &= w^2\phi(w) + w\Phi(w) \\
\frac{d}{dw}\phi(w) &= -w\phi(w)
\end{aligned}
$$

we get

$$
\begin{aligned}
\frac{1}{2\sigma}\int_{\mu-\sigma}^{\mu+\sigma}w^2\phi(w) + w\Phi(w)\,dw &= \frac{1}{4\sigma}\left\{\begin{array}{c}\left[1+(\mu+\sigma)^2\right]\Phi(\mu+\sigma) - \left[\left(1+(\mu-\sigma)^2\right)\right]\Phi(\mu-\sigma) \\ + (\mu-\sigma)\phi(\mu-\sigma) - (\mu+\sigma)\phi(\mu+\sigma)\end{array}\right\} \\
-\frac{(\mu-\sigma)}{2\sigma}\int_{\mu-\sigma}^{\mu+\sigma}w\phi(w)\,dw &= \frac{(\mu-\sigma)}{2\sigma}\left[\phi(\mu+\sigma) - \phi(\mu-\sigma)\right] \\
\int_{\mu+\sigma}^{\infty}w\phi(w)\,dw &= \phi(\mu+\sigma)
\end{aligned}
$$

Summing these terms and rearranging, we get

$$
\mathbb{E}_t[\max(\varepsilon_{t+1}, \bar{\rho})] = \frac{1}{4\sigma}\left\{\begin{array}{c}\left[1+(\mu+\sigma)^2\right]\Phi(\mu+\sigma) - \left[1+(\mu-\sigma)^2\right]\Phi(\mu-\sigma) \\ + (\mu+\sigma)\phi(\mu+\sigma) - (\mu-\sigma)\phi(\mu-\sigma)\end{array}\right\}
$$

which is the result in equation (7) in the paper for $b \equiv \mu + \sigma$ and $a \equiv \mu - \sigma$. In the limit where $\bar{\mu} \to -\infty$, we have $\mu \to -\infty$. In this limit, using properties (i), (iii) and (iv) of lemma 1, we get

$$
\lim_{\mu \to -\infty}\mathbb{E}_t[\max(\varepsilon_{t+1}, \bar{\rho})] = 0 = \mathbb{E}_t(\varepsilon_{t+1})
$$

This result and the fact that $\frac{\partial}{\partial\mu}\mathbb{E}_t[\max(\varepsilon_{t+1}, \bar{\rho})] > 0$ (see equations (18) to (20) below) imply that

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$\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right] > 0$. Finally, in the limit as $\bar{\sigma} \to 0$, we have $\sigma \to 0$ and, using the rule of L'Hôpital,

$$\lim_{\sigma \to 0} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right] = \phi\left(\mu\right) + \mu\Phi\left(\mu\right)$$

When the shadow short rate follows an AR-1 process, this recovers equation (4) in Grisse et al. (2017) for a known but non-zero lower bound.

Next we prove equation (8) in the paper. We have

$$\mathbb{V}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right] = \int_{-\infty}^{+\infty} w^2 f\left(w\right) dw - \left[\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right]\right]^2$$

The first term on the right-hand side is

$$\int_{-\infty}^{+\infty} w^2 f\left(w\right) dw = \frac{1}{2\sigma} \int_{\mu-\sigma}^{\mu+\sigma} \left[w^3\phi\left(w\right) + w^2\Phi\left(w\right) - \left(\mu-\sigma\right)w^2\phi\left(w\right)\right] dw + \int_{\mu+\sigma}^{\infty} w^2\phi\left(w\right) dw$$

Using the fact that

$$\frac{d}{dw}\frac{1}{3}\left[w^3\Phi\left(w\right) - 2\left(2 + w^2\right)\phi\left(w\right)\right] = w^3\phi\left(w\right) + w^2\Phi\left(w\right)$$

$$\frac{d}{dw}\left[\Phi\left(w\right) - w\phi\left(w\right)\right] = w^2\phi\left(w\right)$$

we get

$$\frac{1}{2\sigma}\int_{\mu-\sigma}^{\mu+\sigma} w^3\phi\left(w\right) + w^2\Phi\left(w\right) dw = \frac{1}{6\sigma}\left\{\begin{array}{c}\left(\mu+\sigma\right)^3\Phi\left(\mu+\sigma\right) - \left(\mu-\sigma\right)^3\Phi\left(\mu-\sigma\right) \\ -2\left[2 + \left(\mu+\sigma\right)^2\right]\phi\left(\mu+\sigma\right) + 2\left[2 + \left(\mu-\sigma\right)^2\right]\phi\left(\mu-\sigma\right)\end{array}\right\}$$

$$-\frac{\mu-\sigma}{2\sigma}\int_{\mu-\sigma}^{\mu+\sigma} w^2\phi\left(w\right) dw = \frac{\mu-\sigma}{2\sigma}\left\{\begin{array}{c}\left(\mu+\sigma\right)\phi\left(\mu+\sigma\right) - \left(\mu-\sigma\right)\phi\left(\mu-\sigma\right) \\ -\Phi\left(\mu+\sigma\right) + \Phi\left(\mu-\sigma\right)\end{array}\right\}$$

$$\int_{\mu+\sigma}^{\infty} w^2\phi\left(w\right) dw = 1 + \left(\mu+\sigma\right)\phi\left(\mu+\sigma\right) - \Phi\left(\mu+\sigma\right)$$

where the last line uses property (i) of lemma 1. Summing these terms and rearranging, we get

$$\int_{-\infty}^{+\infty} w^2 f\left(w\right) dw = 1 + \frac{1}{6\sigma}\left\{\begin{array}{c}\left[\left(\mu+\sigma\right)^2 - 3\right]\left(\mu+\sigma\right)\Phi\left(\mu+\sigma\right) - \left[\left(\mu-\sigma\right)^2 - 3\right]\left(\mu-\sigma\right)\Phi\left(\mu-\sigma\right) \\ + \left[\left(\mu+\sigma\right)^2 - 4\right]\phi\left(\mu+\sigma\right) - \left[\left(\mu-\sigma\right)^2 - 4\right]\phi\left(\mu-\sigma\right)\end{array}\right\}$$

which after subtracting the square of the mean leads to the result in equation (8) in the paper. In the limit where $\bar{\mu} \to -\infty$, we have $\mu \to -\infty$. In this limit, using the results from lemma 1, we have

$$\lim_{\mu \to -\infty} \mathbb{V}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right] = \mathbb{V}_t\left(\varepsilon_{t+1}\right) = 1$$

so that $\mathbb{V}_t(r_{t+1}) \to \sigma_\varepsilon^2$ as claimed in the text. ∎

**Proof of proposition 3.** A mean-preserving spread of the lower bound distribution is equivalent to an increase in $\bar{\sigma}$. A mean-preserving contraction therefore lowers $\mathbb{E}_t\left(r_{t+1}\right)$ if

$$\frac{\partial}{\partial\bar{\sigma}}\mathbb{E}_t\left(r_{t+1}\right) = \frac{1}{\sigma_\varepsilon}\frac{\partial}{\partial\sigma}\mathbb{E}_t\left(r_{t+1}\right) = \frac{1}{\sigma_\varepsilon}\frac{\partial}{\partial\sigma}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right] > 0$$

We have

$$\frac{\partial}{\partial\sigma}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right] = \frac{1}{4\sigma^2}\left\{\begin{array}{l}\left(1+\mu^2-\sigma^2\right)\left[\Phi\left(\mu-\sigma\right)-\Phi\left(\mu+\sigma\right)\right] \\ +\left(\mu+\sigma\right)\phi\left(\mu-\sigma\right)-\left(\mu-\sigma\right)\phi\left(\mu+\sigma\right)\end{array}\right\} \tag{12}$$

We will show that this expression is positive by proving four separate results:

$$\lim_{\mu\to-\infty}\frac{\partial}{\partial\sigma}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right] = 0 \tag{13}$$

$$\lim_{\mu\to\infty}\frac{\partial}{\partial\sigma}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right] = 0 \tag{14}$$

$$\frac{\partial^2}{\partial\sigma\partial\mu}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right]\big|_{\mu<0} > 0 \tag{15}$$

$$\frac{\partial^2}{\partial\sigma\partial\mu}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right]\big|_{\mu>0} < 0 \tag{16}$$

Together, these properties imply that the derivative of $\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right]$ with respect to $\sigma$ is always positive: this derivative tends to zero at $\mu\to\pm\infty$, is increasing in $\mu$ for $\mu<0$, has a maximum at $\mu=0$ and is decreasing in $\mu$ for $\mu>0$. We now prove (13) to (16) in turn.

For (13), we have

$$\lim_{\mu\to-\infty}\frac{\partial}{\partial\sigma}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right] = \frac{1}{4\sigma^2}\left\{\begin{array}{l}\mu^2\Phi\left(\mu-\sigma\right)-\mu^2\Phi\left(\mu+\sigma\right) \\ +\mu\phi\left(\mu-\sigma\right)-\mu\phi\left(\mu+\sigma\right)\end{array}\right\} = 0$$

by lemma 1, as each term individually tends to zero.

Next, for (14) we have

$$\begin{aligned}\lim_{\mu\to\infty}\frac{\partial}{\partial\sigma}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right] &= \frac{1}{4\sigma^2}\left\{\begin{array}{l}\mu^2\left[\Phi\left(\mu-\sigma\right)-\Phi\left(\mu+\sigma\right)\right] \\ +\mu\phi\left(\mu-\sigma\right)-\mu\phi\left(\mu+\sigma\right)\end{array}\right\} \\ &= \frac{1}{4\sigma^2}\mu^2\left[\Phi\left(\mu-\sigma\right)-\Phi\left(\mu+\sigma\right)\right]\end{aligned}$$

by lemma 1. Now using the rule of L'Hôpital, we get

$$\begin{aligned}\lim_{\mu\to\infty}\frac{\partial}{\partial\sigma}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar\rho\right)\right] &= \lim_{\mu\to\infty}\frac{1}{4\sigma^2}\frac{\Phi\left(\mu-\sigma\right)-\Phi\left(\mu+\sigma\right)}{\frac{1}{\mu^2}} \\ &= \lim_{\mu\to\infty}\frac{1}{4\sigma^2}\frac{\frac{d}{d\mu}\left[\Phi\left(\mu-\sigma\right)-\Phi\left(\mu+\sigma\right)\right]}{\frac{d}{d\mu}\frac{1}{\mu^2}} \\ &= -\lim_{\mu\to\infty}\frac{\mu^3}{8\sigma^2}\left[\phi\left(\mu-\sigma\right)-\phi\left(\mu+\sigma\right)\right]\end{aligned}$$

Now

$$\lim_{\mu\to\infty}\mu^3\phi\left(\mu-\sigma\right) = \lim_{\mu\to\infty}\mu^3\phi\left(\mu+\sigma\right) = \lim_{\mu\to\infty}\mu^3\phi\left(\mu\right)$$

Using the rule of L'Hôpital repeatedly, we get

$$
\begin{aligned}
\lim_{\mu \to \infty} \mu^3 \phi(\mu) &= \frac{1}{\sqrt{2\pi}} \lim_{\mu \to \infty} \frac{\mu^3}{e^{\frac{1}{2}\mu^2}} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\mu \to \infty} \frac{\frac{d}{d\mu}\left(\mu^3\right)}{\frac{d}{d\mu}\left(e^{\frac{1}{2}\mu^2}\right)} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\mu \to \infty} \frac{3\mu}{e^{\frac{1}{2}\mu^2}} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\mu \to \infty} \frac{3}{\mu e^{\frac{1}{2}\mu^2}} \\
&= 0
\end{aligned}
$$

It follows that $\lim_{\mu \to \infty} \frac{\partial}{\partial \sigma} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] = 0 - 0 = 0$ which concludes the proof of property (14).

Next, we consider properties (15) and (16). We have

$$
\frac{\partial^2}{\partial \sigma \partial \mu} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] = \frac{1}{2\sigma^2} \left\{ \begin{array}{c} \mu\left[\Phi(\mu - \sigma) - \Phi(\mu + \sigma)\right] \\ + \phi(\mu - \sigma) - \phi(\mu + \sigma) \end{array} \right\}
$$

We want to show that this expression is positive for $\mu < 0$ and negative for $\mu > 0$. It is sufficient to show how the expression in curly brackets varies with $\sigma$. In the limit where $\sigma \to 0$ we have

$$
\lim_{\sigma \to 0} \left\{ \begin{array}{c} \mu\left[\Phi(\mu - \sigma) - \Phi(\mu + \sigma)\right] \\ + \left[\phi(\mu - \sigma) - \phi(\mu + \sigma)\right] \end{array} \right\} = 0
$$

Next, the derivative with respect to $\sigma$ is

$$
\frac{\partial}{\partial \sigma} \left\{ \begin{array}{c} \mu\left[\Phi(\mu - \sigma) - \Phi(\mu + \sigma)\right] \\ + \left[\phi(\mu - \sigma) - \phi(\mu + \sigma)\right] \end{array} \right\} = \sigma\left[\phi(\mu + \sigma) - \phi(\mu - \sigma)\right]
$$

This expression is positive if $\mu < 0$, negative if $\mu > 0$, and zero if $\mu = 0$.

It follows that $\frac{\partial^2}{\partial \sigma \partial \mu} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right]$ has the following properties. First, as claimed in (15), the expression is positive for $\mu < 0$. This follows from the fact that it approaches zero as $\sigma \to 0$ and is increasing in $\sigma$ for $\mu < 0$. Second, as claimed in (16), the expression is negative for $\mu > 0$. This follows from the fact that it is decreasing in $\sigma$ for $\mu > 0$ and approaches zero as $\sigma \to 0$. Third, for $\mu = 0$ we have $\frac{\partial^2}{\partial \sigma \partial \mu} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] = 0$.

Putting everything together, properties (13) to (16) imply that

$$
\frac{\partial}{\partial \sigma} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] > 0
$$

and therefore that

$$
\frac{\partial}{\partial \bar{\sigma}} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] = \frac{\partial \sigma}{\partial \bar{\sigma}} \times \frac{\partial}{\partial \sigma} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] = \frac{1}{\sigma_\varepsilon} \frac{\partial}{\partial \sigma} \mathbb{E}_t\left[\max\left(\varepsilon_{t+1}, \bar{\rho}\right)\right] > 0
$$

as required. ∎

**Proof of proposition 4.** We have

$$
\begin{aligned}
\frac{\partial \mathbb{E}_t \left( r_{t+1} \right)}{\partial r_t} &= \frac{\partial \mathbb{E}_t \left( r_{t+1}^* \right)}{\partial r_t} + \sigma_\varepsilon \frac{\partial \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right]}{\partial \mu} \frac{\partial \mu}{\partial \mathbb{E}_t \left( r_{t+1}^* \right)} \frac{\partial \mathbb{E}_t \left( r_{t+1}^* \right)}{\partial r_t} \\
&= \frac{\partial \mathbb{E}_t \left( r_{t+1}^* \right)}{\partial r_t} \left[ 1 - \frac{\partial \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right]}{\partial \mu} \right]
\end{aligned}
$$

We first show that this is positive, as claimed in the text. With $\partial \mathbb{E}_t \left( r_{t+1}^* \right) / \partial r_t > 0$ by assumption, we only need to show that

$$
\frac{\partial \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right]}{\partial \mu} = \frac{1}{2\sigma} \left\{ \begin{array}{c} \left( \mu + \sigma \right) \Phi \left( \mu + \sigma \right) - \left( \mu - \sigma \right) \Phi \left( \mu - \sigma \right) \\ + \phi \left( \mu + \sigma \right) - \phi \left( \mu - \sigma \right) \end{array} \right\} < 1 \tag{17}
$$

where we have used equation (7) in the paper. Using lemma 1, we have (i)

$$
\lim_{\mu \to -\infty} \frac{\partial}{\partial \mu} \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right] = 0 \tag{18}
$$

since each term in curly brackets individually approaches zero. We also have (ii)

$$
\frac{\partial^2}{\partial \mu^2} \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right] = \frac{1}{2\sigma} \left[ \Phi \left( \mu + \sigma \right) - \Phi \left( \mu - \sigma \right) \right] > 0 \tag{19}
$$

since $\sigma > 0$. Properties (i) and (ii) together imply that

$$
\frac{\partial}{\partial \mu} \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right] > 0 \tag{20}
$$

The limit as $\mu \to +\infty$ is

$$
\begin{aligned}
\lim_{\mu \to \infty} \frac{\partial \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right]}{\partial \mu} &= \lim_{\mu \to \infty} \frac{1}{2\sigma} \left\{ \begin{array}{c} \sigma \left[ \Phi \left( \mu + \sigma \right) + \Phi \left( \mu - \sigma \right) \right] \\ + \mu \left[ \Phi \left( \mu + \sigma \right) - \Phi \left( \mu - \sigma \right) \right] \end{array} \right\} \\
&= 1 + \frac{1}{2\sigma} \lim_{\mu \to \infty} \frac{\frac{\partial}{\partial \mu} \left[ \Phi \left( \mu + \sigma \right) - \Phi \left( \mu - \sigma \right) \right]}{\frac{\partial}{\partial \mu} \frac{1}{\mu}} \\
&= 1 - \frac{1}{2\sigma} \lim_{\mu \to \infty} \mu^2 \left[ \phi \left( \mu + \sigma \right) - \phi \left( \mu - \sigma \right) \right] \\
&= 1
\end{aligned}
$$

where the second line uses the rule of L'Hôpital and the fourth line result (ii) from lemma 1. It follows that $\partial \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right] / \partial \mu$ is between zero and one, and hence that $\partial \mathbb{E}_t \left( r_{t+1} \right) / \partial r_t > 0$ as claimed.

Now since $\partial \mathbb{E}_t \left( r_{t+1}^* \right) / \partial r_t > 0$ by assumption (and $\mathbb{E}_t \left( r_{t+1}^* \right)$ does not depend on $\bar{\sigma}$) we have

$$
\begin{aligned}
\frac{\partial^2 \mathbb{E}_t \left( r_{t+1} \right)}{\partial r_t \partial \bar{\sigma}} &= -\frac{\partial \mathbb{E}_t \left( r_{t+1}^* \right)}{\partial r_t} \frac{\partial^2 \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right]}{\partial \mu \partial \sigma} \times \frac{\partial \sigma}{\partial \bar{\sigma}} \\
&= -\frac{\partial \mathbb{E}_t \left( r_{t+1}^* \right)}{\partial r_t} \frac{\partial^2 \mathbb{E}_t \left[ \max \left( \varepsilon_{t+1}, \bar{\rho} \right) \right]}{\partial \mu \partial \sigma} \times \frac{1}{\sigma_\varepsilon}
\end{aligned}
$$

Using results (15) to (16) it follows that this expression is positive if $\mu > 0$ and negative if $\mu < 0$. The expression equals zero when $\mu = 0$. ∎
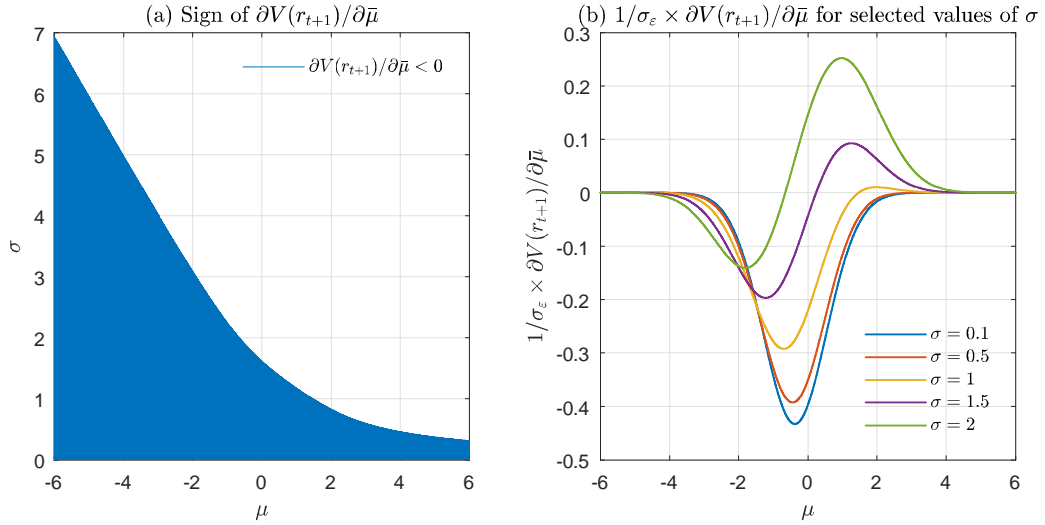
Figure 3: Effect of a change in the mean of $\bar{r}$ on $\frac{1}{\sigma_\varepsilon}\mathbb{V}_t\left(r_{t+1}\right)$. An increase in $\mu$ corresponds to a higher market-perceived probability that the lower bound is binding.

# B   Additional results: effects of a decline in the expected lower bound

The model in section 2.3 of the paper also lends itself to analyzing the effects of a change in the expected lower bound $\bar{\mu}$. In a setting with a known lower bound, Grisse et al. (2017) emphasized that expected future short-term interest rates decline in response to an unexpected and permanent change in the lower bound. This result naturally carries over to the case where $\bar{r}$ is uncertain, and its whole distribution shifts downward. In particular, using (20) we have

$$\frac{\partial}{\partial\bar{\mu}}\mathbb{E}_t\left(r_{t+1}\right) = \sigma_\varepsilon\frac{\partial}{\partial\bar{\mu}}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right] = \sigma_\varepsilon\frac{\partial\mu}{\partial\bar{\mu}}\times\frac{\partial}{\partial\mu}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right] = \frac{\partial}{\partial\mu}\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right] > 0$$

In the context of a known lower bound, Grisse et al. (2017) also show that as the lower bound becomes increasingly binding – as $\bar{\mu}$ increases – changes in short-term rates elicit smaller responses in longer-term bond yields. This finding also carries over to the case where the lower bound is uncertain. Since $\partial^2\mathbb{E}_t\left(r_{t+1}^*\right)/\partial r_t\partial\bar{\mu} = 0$ and $\partial\mathbb{E}_t\left(r_{t+1}^*\right)/\partial r_t > 0$ we have

$$
\begin{aligned}
\frac{\partial^2\mathbb{E}_t\left(r_{t+1}\right)}{\partial r_t\partial\bar{\mu}} &= -\frac{\partial\mathbb{E}_t\left(r_{t+1}^*\right)}{\partial r_t}\frac{\partial^2\mathbb{E}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right]}{\partial^2\mu}\times\frac{\partial\mu}{\partial\bar{\mu}} \\
&= -\frac{\partial\mathbb{E}_t\left(r_{t+1}^*\right)}{\partial r_t}\frac{1}{2\sigma}\left[\Phi\left(\mu+\sigma\right)-\Phi\left(\mu-\sigma\right)\right]\times\frac{1}{\sigma_\varepsilon} < 0
\end{aligned}
$$

How does a change in the expected lower bound influence the variance of expected future short-term interest rates, and hence the term premium? We have

$$\frac{\partial\mathbb{V}_t\left(r_{t+1}\right)}{\partial\bar{\mu}} = \sigma_\varepsilon^2\frac{\partial\mathbb{V}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right]}{\partial\bar{\mu}}\frac{\partial\mu}{\partial\bar{\mu}} = \sigma_\varepsilon\frac{\partial\mathbb{V}_t\left[\max\left(\varepsilon_{t+1},\bar{\rho}\right)\right]}{\partial\mu}$$

Figure 3 shows how the sign and magnitude of $\partial\mathbb{V}_t\left(r_{t+1}\right)/\partial\bar{\mu}$ depends on $\mu$ and $\sigma$. Panel (a) shows that $\partial\mathbb{V}_t\left(r_{t+1}\right)/\partial\bar{\mu}$ is negative if $\mu$ and $\sigma$ are sufficiently small, i.e., if the lower bound is binding with a

sufficiently small probability and lower bound uncertainty is sufficiently small relative to the uncertainty around the shadow short rate. Panel (b) shows how $\partial \mathbb{V}_t \left( r_{t+1} \right) / \partial \bar{\mu}$ varies with $\mu$, for selected values of $\sigma$. As $\mu \to -\infty$, the expected shadow short rate is well above the lower bound, so that the lower bound becomes quantitatively irrelevant. In this limit, $\partial \mathbb{V}_t \left( r_{t+1} \right) / \partial \bar{\mu} \to 0$ since $\mathbb{V}_t \left( r_{t+1} \right) \to \mathbb{V}_t \left( r_{t+1}^* \right)$. As $\mu$ increases from very low levels, $\mathbb{V}_t \left( r_{t+1} \right)$ initially falls, since the range of possible values for the short-term interest rate declines. As $\mu$ increases further and the likelihood that the lower bound will be binding increases, $\mathbb{V}_t \left( r_{t+1} \right)$ starts to increase. In the limit as $\mu \to +\infty$, the short rate is entirely determined by the lower bound, such that $\mathbb{V}_t \left( r_{t+1} \right) \to \mathbb{V}_t \left( \bar{r} \right)$ which is again unaffected by $\bar{\mu}$. The strength of these effects varies with $\sigma$. For very low values of $\sigma$, lower bound uncertainty is small relative to uncertainty about the shadow short rate. In this case, $\partial \mathbb{V}_t \left( r_{t+1} \right) / \partial \bar{\mu}$ is always negative: as $\mu$ increases and the lower bound becomes increasingly binding, the variance of the short-term interest rate falls. As $\sigma$ increases, lower bound uncertainty becomes larger relative to uncertainty about the shadow short rate. In this case, $\mathbb{V}_t \left( r_{t+1} \right)$ increases with $\bar{\mu}$ if the likelihood that the lower bound will bind ($\mu$) is sufficiently high.

In most situations where the lower bound is quantitatively relevant $\mu$ is likely to be negative but small in absolute value, with shadow short rates expected to be not much lower than the mean of $\bar{r}$. At the same time, shadow rate uncertainty $\mathbb{V}_t \left( r_{t+1}^* \right) = \sigma_\varepsilon^2$ likely dominates the uncertainty around the lower bound $\mathbb{V}_t \left( \bar{r} \right) = \bar{\sigma}^2 / 3$, so that $\sigma < \sqrt{3} \approx 1.7$. Figure 3 shows that in this environment, the variance of future short rates and hence the term premium will decrease if $\bar{\mu}$ increases.

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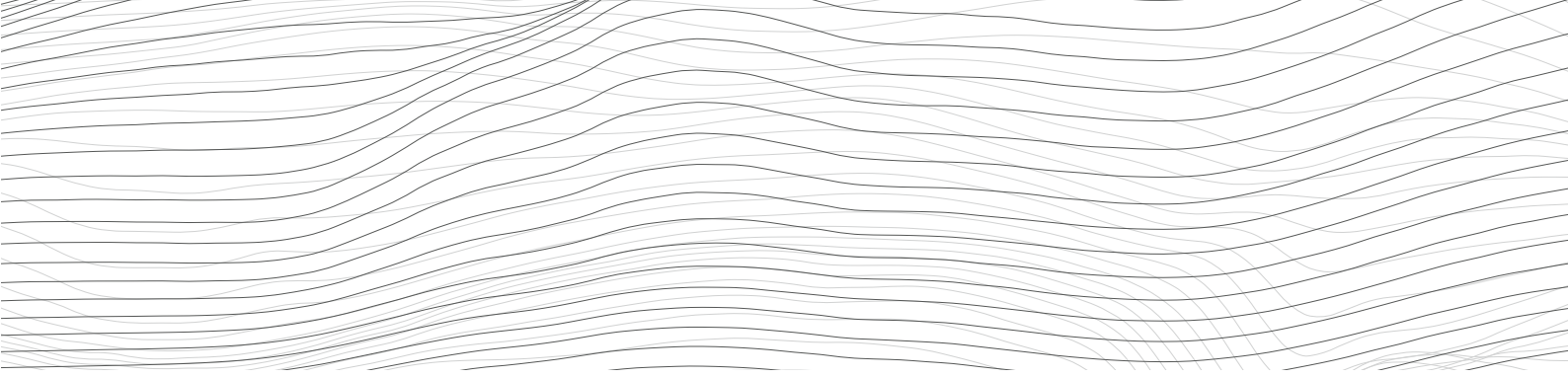
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